are fragments which the sun has picked up and attached to our system at some epoch in the sun's very ancient history. They are certainly fragments from a once larger body—thereby testifying to the fact that large solid bodies are sometimes broken up and their remains scattered. It seems certain that there must be more or less of this material in space.

How much of this material will the sun and planets pick up as they move along? It will simplify matters to suppose that this material is initially stationary and that the particles move only under the sun's gravitation. This hypothesis is doubtless contrary to the facts but inasmuch as their motions will be at random with respect to the sun this hypothesis will give at least a first approximation to the actual facts. The particles, then, will describe hyperbolas about the sun, and every particle whose perihelion distance is less than the radius of the sun will fall into the sun and add its mass to the sun's mass. All of the others will pass by and be of no consequence.

Imagine the sun moving along a straight line with a speed s, and consider a particle at a distance D from this line while the sun is still very remote. Since the particle is initially at rest it has a speed s with respect to the sun along a line parallel to the sun's motion. If k^2 is the gravitational constant, M the sun's mass and a the semi-axis of the particles' orbit about the sun, then the velocity of the particle with respect to the sun at a distance r is given by the formula

$$v^2 = k^2 M\left(\frac{2}{r} + \frac{1}{a}\right).$$

When $r = \infty$ then $v = s$, from which it follows that

$$a = \frac{k^2 M}{s^2}. \tag{1}$$

From the law of areas, or moment of momentum, we have also

$$xy' - yx' = \sqrt{k^2 Ma(\epsilon^2 - 1)},$$

where ϵ is the eccentricity of the hyperbolic orbit. But when very remote from the sun $y' = 0$, $y = D$, $x' = -s$, and therefore

$$D^2 s^2 = k^2 Ma(\epsilon^2 - 1). \tag{2}$$

Eliminating a between (1) and (2) and then solving for ϵ we have

$$\epsilon = \sqrt{1 + \frac{s^4 D^2}{k^4 M^2}}, \tag{3}$$

so that the perihelion distance is

$$a(\epsilon - 1) = \sqrt{\frac{k^4 M^2}{s^4} + D^2} - \frac{k^2 M}{s^2}.$$

If now the radius of the sun is R, then in order that the particle may strike the sun we must have

$$\sqrt{\frac{k^4M^2}{s^4}+D^2}-\frac{k^2M}{s^2}<R,$$

from which it is seen that

$$D^2<\frac{2k^2MR}{s^2}+R^2. \tag{4}$$

In the case of a body having the mass, speed, and size of the sun the term R^2 is small as compared to $2k^2MR/s^2$, so that if we take a cylinder of radius

$$\rho=\frac{\sqrt{2k^2MR}}{s}, \tag{5}$$

we can say that all of the particles within this cylinder will be swept up by the sun, and the sun's *effective* radius is ρ. For the sun its numerical value is 14,000,000 miles.

Since in one unit of time the sun moves through a distance s, it follows that the rate at which material is swept up is

$$\frac{dM}{dt}=\pi\rho^2s\delta=\pi\delta\left(\frac{2k^2MR}{s}+R^2s\right), \tag{6}$$

where δ is the average density of matter in space. This expression has a minimum for $s^2=2k^2M/R$, which for our sun gives $s=383$ miles per second, and for this minimum value $\rho=\sqrt{2}R$. For such speeds as that of the sun the last term of (6) might be neglected in which case we would have the result that the rate at which the mass increases is inversely proportional to the speed.

If we knew the value of δ, which is the amount of matter in a unit volume of space we could compute the rate at which the sun is growing. Without doubt δ varies from one region of space to another. In a recent article in the *Astrophysical Journal*[1] the estimate was made that one particle one one-hundredth of an inch in diameter for every 560 cubic miles of space would account for the blackness of the night sky. Taking this value of the density and the present speed of the sun it would require 1.4×10^{17} years for the mass of the sun to double. If, however, the sun should enter a nebulous region in which the density is as low as 10^{-12} times the density of air at sea level its mass would double in one billion years. A density of this tenuity would be obtained if one cubic foot of air were so expanded as to fill 8 cubic miles. We have of course no means of knowing what the densities of the nebulas are but it is evident that if the sun should penetrate such a region as that of the Orion Nebula its mass would be materially increased in a relatively short time. But independently of such special regions it is clear that not only the sun but all of the stars are gathering in material unto themselves sometimes rapidly but usually slowly.

[1] MacMillan, *Astrophysical Journal*, volume 48, July, 1918.

A closely related subject is the effect of this process upon the planetary system. In order to simplify the discussion let us suppose that the sun and Jupiter in their Keplerian motion pass through a region of space filled with material which they gather in as they journey along, and seek the effect of this growth upon the system. Let ξ_1, η_1, ζ_1 and ξ_2, η_2, ζ_2 be the coördinates of Jupiter and the sun respectively with respect to axes fixed in space; m_1 and m_2 are their masses. The components of momentum at any instant will be $m_1\xi_1'$, $m_1\eta_1'$, $m_1\zeta_1'$; $m_2\xi_2'$, $m_2\eta_2'$, $m_2\zeta_2'$. If either or both of the bodies collide with a stationary mass or particle their masses will be increased and their velocities diminished, but their instantaneous momenta will be unaltered. The differential relations between the changes of velocities and masses are therefore

$$
\begin{aligned}
&d\xi_1' = -\frac{\xi_1'}{m_1}dm_1, \qquad d\eta_1' = -\frac{\eta_1'}{m_1}dm_1, \qquad d\zeta_1' = -\frac{\zeta_1'}{m_1}dm_1,\\
&d\xi_2' = -\frac{\xi_2'}{m_2}dm_2, \qquad d\eta_2' = -\frac{\eta_2'}{m_2}dm_2, \qquad d\zeta_2' = -\frac{\zeta_2'}{m_2}dm_2.
\end{aligned} \tag{7}
$$

If $\bar{\xi}, \bar{\eta}, \bar{\zeta}$ are the coördinates of the center of mass and M is the sum of the masses, the integrals for the motion of the center of mass are

$$
\begin{aligned}
M\bar{\xi}' &= m_1\xi_1' + m_2\xi_2' = \alpha_1 = Ms\cos\lambda_1,\\
M\bar{\eta}' &= m_1\eta_1' + m_2\eta_2' = \alpha_2 = Ms\cos\lambda_2,\\
M\bar{\zeta}' &= m_1\zeta_1' + m_2\zeta_2' = \alpha_3 = Ms\cos\lambda_3,
\end{aligned} \tag{8}
$$

where s is the speed of the center of gravity and $\lambda_1, \lambda_2, \lambda_3$ are the direction angles of the path of the center of gravity. It is evident that α_1, α_2, and α_3 are unaltered by the collision.

We will take a new set of axes parallel to the fixed axes but with the origin at the sun. Then the coördinates of Jupiter with respect to the sun will be

$$
\begin{aligned}
x &= \xi_1 - \xi_2, \qquad y = \eta_1 - \eta_2, \qquad z = \zeta_1 - \zeta_2,\\
x' &= \xi_1' - \xi_2', \qquad y' = \eta_1' - \eta_2', \qquad z' = \zeta_1' - \zeta_2'.
\end{aligned} \tag{9}
$$

By differentiation of the second set of (9) we get

$$dx' = d\xi_1' - d\xi_2', \qquad dy' = d\eta_1' - d\eta_2', \qquad dz' = d\zeta_1' - d\zeta_2';$$

and by substitution of values from (7)

$$
\begin{aligned}
dx' &= -\left[\frac{\xi_1'}{m_1}dm_1 - \frac{\xi_2'}{m_2}dm_2\right],\\
dy' &= -\left[\frac{\eta_1'}{m_1}dm_1 - \frac{\eta_2'}{m_2}dm_2\right],\\
dz' &= -\left[\frac{\zeta_1'}{m_1}dm_1 - \frac{\zeta_2'}{m_2}dm_2\right].
\end{aligned} \tag{10}
$$

After solving (8) and (9) for $\xi_1', \eta_1', \zeta_1'$; $\xi_2', \eta_2', \zeta_2'$ and substituting in (10) we have

$$dx' = -\frac{\alpha_1}{M}\left(\frac{dm_1}{m_1} - \frac{dm_2}{m_2}\right) - \frac{x'}{M}\left(\frac{m_2}{m_1}dm_1 + \frac{m_1}{m_2}dm_2\right),$$

$$(11) \qquad dy' = -\frac{\alpha_2}{M}\left(\frac{dm_1}{m_1} - \frac{dm_2}{m_2}\right) - \frac{y'}{M}\left(\frac{m_2}{m_1}dm_1 + \frac{m_1}{m_2}dm_2\right),$$

$$dz' = -\frac{\alpha_3}{M}\left(\frac{dm_1}{m_1} - \frac{dm_2}{m_2}\right) - \frac{z'}{M}\left(\frac{m_2}{m_1}dm_1 + \frac{m_1}{m_2}dm_2\right).$$

The change in the relative velocity $v^2 = x'^2 + y'^2 + z'^2$ will be

$$vdv = x'dx' + y'dy' + z'dz',$$

and by means of (11) this expression becomes

$$(12) \quad vdv = -\frac{1}{M}(\alpha_1 x' + \alpha_2 y' + \alpha_3 z')\left(\frac{dm_1}{m_1} - \frac{dm_2}{m_2}\right) - \frac{v^2}{M}\left(\frac{m_2}{m_1}dm_1 + \frac{m_1}{m_2}dm_2\right).$$

If μ_1, μ_2, μ_3 are the direction angles of v we have

$$x' = v\cos\mu_1, \qquad y' = v\cos\mu_2, \qquad z' = v\cos\mu_3,$$

$$\alpha_1 = Ms\cos\lambda_1, \qquad \alpha_2 = Ms\cos\lambda_2, \qquad \alpha_3 = Ms\cos\lambda_3;$$

and if ω is the angle between the direction of motion of the center of gravity and the direction of the relative motion of Jupiter we have

$$\cos\omega = \cos\lambda_1\cos\mu_1 + \cos\lambda_2\cos\mu_2 + \cos\lambda_3\cos\mu_3$$

and therefore, on removing a factor v, equation (12) becomes

$$(13) \qquad dv = -s\cos\omega\left(\frac{dm_1}{m_1} - \frac{dm_2}{m_2}\right) - \frac{v}{M}\left(\frac{m_2}{m_1}dm_1 + \frac{m_1}{m_2}dm_2\right).$$

If the direction of motion of the center of gravity were perpendicular to the plane of relative motion the $\cos\omega$ would always be zero. If it is not perpendicular then $\cos\omega$ varies, as Jupiter completes a circuit about the sun, from a certain positive value to the same negative value; the extreme limits being ± 1 when the direction of motion of the center of gravity lies in the plane of relative motion. If we suppose the added material to be gathered in slowly at all points of the orbit the average value of $\cos\omega$ will be zero, and in the long run this term will be inappreciable. Furthermore it would be exactly zero if the material gathered in by the sun and Jupiter was proportional to their own masses. This term gives the change in relative velocity due to a displacement of the center of gravity. Under the conditions we are discussing its effect is very small and it will be neglected. It is found then that (13) can be written

$$\frac{dv}{v} = \frac{1}{\frac{1}{m_1} + \frac{1}{m_2}}d\left(\frac{1}{m_1} + \frac{1}{m_2}\right),$$

so that if A is the constant of integration we have

$$v = A\left(\frac{1}{m_1}+\frac{1}{m_2}\right).$$

If we suppose that in the accretion of material the ratio of m_1 to m_2 is not altered we will have

$$Mv = B, \tag{14}$$

where B is a constant not altered by the accretion.

The expression for v in the relative motion is

$$v^2 = k^2M\left(\frac{2}{r}-\frac{1}{a}\right).$$

On multiplying through by M^2 we have

$$B^2 = k^2M^3\left(\frac{2}{r}-\frac{1}{a}\right).$$

If a collision occurs r is not altered, nor is B. Hence by differentiation with respect to M and a

$$3M^2\left(\frac{2}{r}-\frac{1}{a}\right)dM+\frac{M^3}{a^2}\,da = 0,$$

which shows how the major axis is affected by the collision. If the process of accretion is a very slow one as we suppose is the case so that all parts of the orbit share alike in the collisions then we can replace $1/r$ by its average value, viz.,[1] $1/a$, after which this differential relation reduces to

$$3\frac{dM}{M}+\frac{da}{a} = 0,$$

from which it follows that

$$M^3a = C, \quad \text{or} \quad a = \frac{C}{M^3}, \tag{15}$$

where C is a constant not altered by accretion. The expression for the period $P = 2\pi a^{3/2}/k\sqrt{M}$ becomes

$$P = \frac{2\pi C^{3/2}}{kM^5}. \tag{16}$$

Stated in words these results are that during a process of slow growth in the sun and Jupiter due to stationary material picked up by them in their journey through space, their mean distance apart varies inversely as the cube of the sum of their masses, and their period varies inversely as the fifth power of the sum of their masses.

---

[1] The average value of $\frac{1}{r}$ for a complete revolution is $\frac{1}{P}\int_0^P\frac{dt}{r} = \frac{1}{a}$.

For example, suppose the masses of the sun and Jupiter were each increased to 5 times their present mass, then we would have $P_1 = 3125P_2$. That is the period of Jupiter, 11.86 years, would be reduced to $33\frac{1}{4}$ hours, and its distance would be reduced from 500 millions to 4 millions of miles. It is needless to discuss the fate of the terrestrial planets.

If it is true that the energies of the sun and stars are maintained by the subatomic energies of this added material and if the yielding up of its subatomic energies implies a loss in mass, then it is not necessary that the mass of the sun be increasing even though it is gathering in new material. It may even be declining at the present time. Since Jupiter is not wasting much energy in radiation it would seem as though the mass of Jupiter must certainly be growing. It requires but little imagination to see the possibility of Jupiter eventually rivalling the sun in mass and even brilliancy, and the two together constituting a double star.

To be sure, such a conception involves an enormous range in time, but it will never be possible to understand the astronomical forms which are now presented to our vision in the wide expanses of the heavens until we not only understand the physical processes now at work but also extend their logical consequences to such intervals of time as are necessary to establish a cycle, and undoubtedly such an interval is very great.

CUSPIDAL ROSETTES.[1]

By WILLIAM FRANCIS RIGGE, Creighton University, Omaha, Neb.

The rose, rosette, rosace, Rosenkurve or multifolium, or whatever other name it may have, is a periodic polar curve whose equal sectors may have any angular magnitude. Its general equation, as usually given, $\rho = a + b \sin n\theta$,[2] supposes the tracing point to move with a simple harmonic motion of n cycles along a radial line through the pole, at the same time that it makes one revolution about this pole with uniform angular speed. A simple instance of such a polar curve is the trifolium, Fig. 1 (p. 324), which is drawn by having a tracing pen move with simple harmonic motion of amplitude b in a radial line over a uniformly rotating disk in such a way that the pen just touches its centre without passing beyond, and the disk makes one revolution in three cycles of the pen. The equation is then $\rho = a(1 - \sin 3\theta)$, as is seen by inspection in Fig. 2.

If in the general equation a is greater than b, the pen does not reach the center as in Fig. 3, and if a is less than b, the pen passes beyond it as in Fig. 4 and draws

[1] See "Concerning a new Method of Tracing Cardioids" by William F. Rigge, in the January, 1919, MONTHLY and "On the Construction of Certain Curves Given in Polar Coördinates" by R. E. Moritz, in the May, 1917, MONTHLY.

[2] The two terms may have unlike signs and sin be replaced by cos.

smaller lobes or loops on the other side, which are within or between the larger ones according as n is odd or even, until when $a = 0$, the equation becomes $\rho = b \sin 3\theta$, the loops are equal and we have a trifolium in Fig. 5 somewhat like Fig. 1 but half its size and traced twice by the pen because n is odd, the number of the lobes being doubled when n is even. The rosette is *cuspidal* only when $a = b$ as in Figs. 1 and 2, because only then the pen, after tracing one branch of the curve and coming to a momentary standstill, retreats along another branch, so that both branches have a common rectilinear tangent at the point of rest. This tangent is always between the two branches in the curves treated in this discussion, so that the cusps are of the first species.

All this is well known and has been mentioned merely as an introduction to the subject in hand. As Moritz (l. c.) has very completely treated the case of the pen moving along a *radial* line through the center of the disk, the problem now is to investigate what happens when the pen moves along a *non-radial* line, and when it is set down at any initial phase. One who has tried the experiment will know that he obtained a rosette distorted and somewhat like one of the three mentioned before in Figs. 3, 2 and 4, which we may call rounded or curtate, cuspidal, and looped or prolate. Of these the cuspidal rosettes require certain conditions which it is the specific object of the present article to study.

The Cardioid.—As the cardioid may be defined as a cuspidal rosette with the ratio $n = 1$, the present investigations are based upon the previously cited article by the writer, to which the reader must have recourse for more detailed explanations. The following condensed resumé may however be sufficient.

In Fig. 6, A is the center of the disk which rotates with uniform angular speed in a clockwise direction. B is the point at which the tracing pen is set down in any initial phase of its rectilinear simple harmonic motion which has the same period as the disk, so that, if the disk did not rotate, the pen would move over the line ERG, its distance from R, its middle point, being at any moment the sine of the phase.

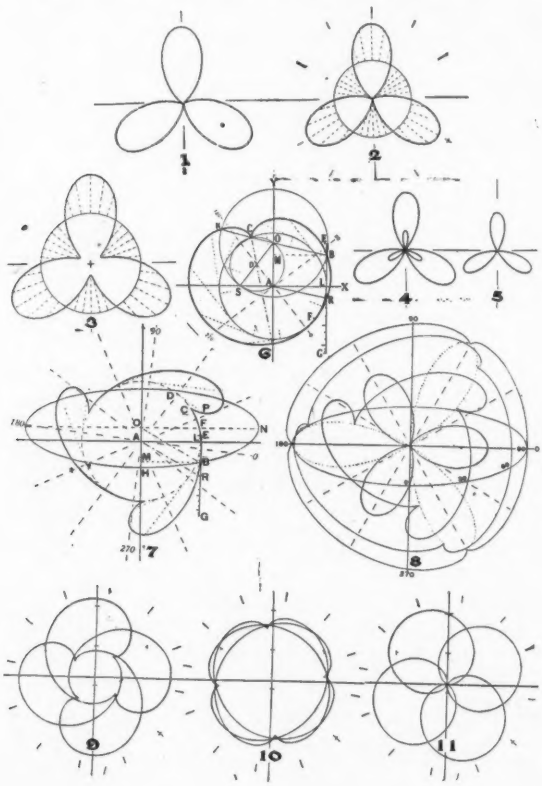
In order to draw a cardioid it is essential that the point B, at which the pen is set down on the rotating disk, should be on the "starting circle" whose radius OB is taken as unity and is equal to the amplitude of the harmonic motion ER or RG, and whose center O is on the Y axis at the distance from the center of the disk equal to $OA = BR = \sin \alpha =$ the sine of the initial phase. The cusp C of the cardioid will then be on the cusp circle, with radius one half of OB, which is internally tangent to the starting circle at S on the axis of X so that the angle $OSA = \alpha$, and which passes through O and A. The selection of the point B on the starting circle fixes the starting angle $\beta = AOB$, which then locates the cusp C on the diameter through B. The distance of the cusp C from A is equal to $AC = AL = MB = \sin \beta$. There is also a second starting point K, diametrically opposite B, so that $\beta' = 180° + \beta$, upon which the pen may be set down at the same initial phase α and trace the same cardioid.

The mechanical method mentioned of tracing the cardioid is what we might call the *tangent* method.[1] A similar method will be seen to apply in a more gen-

[1] Cf. "Concerning a new Method, etc."

eralized form to cuspidal rosettes of all kinds. To facilitate the comparison, the same letters and symbols will be used as much as possible.

Cuspidal Rosettes: the Ratio $n = 3$.—We begin with a simple ratio of $n = 3$, or $n = p/q = 3/1$, that is, p or 3 cycles of the rectilinear motion of the pen being equal to q or one of the disk, Fig. 7. It is immaterial whether we suppose both the pen and the disk to move in the manner indicated, or imagine the disk at rest and credit the pen with its motion in the reverse direction. The initial phase α of the rectilinear motion in Fig. 7 has been taken as 20°, so that



$BR = OA = \sin 20°$. We will take $\beta = 30°$. In comparing Figs. 6 and 7 we may note generic similarities and obvious differences. The starting point B and the rectilinear path ERG are the same in both figures, although the scale is not. The starting circle with radius unity in Fig. 6 becomes the large ellipse in Fig. 7, whose center O is as before on the Y axis at the distance $\sin \alpha$ from A. The conjugate axis of the ellipse OH is unity, and its transverse axis $ON = n = 3$, and is parallel to the X axis. AL is now $n \sin \beta = 3 \sin \beta$.

The Starting Point B.—That the point B must be on the ellipse mentioned may be proved by substituting the general ratio n for its value of unity in the cardioid. If in Fig. 6 we draw BO parallel to RA, these lines are equal, and $OA = BR = \sin \alpha$. Drawing BM parallel to AL makes them also equal to one another, and OM equal to LR. As LR is the sine of some phase of the harmonic motion, we may take it as the cosine of the angle ARL or its equal MOB, that is, as $\cos \beta$. Then $MB = AL = \sin \beta$ and $OB = AR = OS =$ unity. Therefore the starting point B is on the circle with radius unity and with its center O the distance $\sin \alpha$ from A.

In the case of a rosette as in Fig. 7, we have MB also equal to AL, $OA = BR = \sin \alpha$ and $LR = OM$. Taking these last equal to $\cos \beta$ as before, we cannot now make AL and MB equal to $\sin \beta$, but must take them equal to $n \sin \beta$, because the variation of the phase of the pen along the line ERG is now n times the angular speed of the disk. Hence the starting point B is on the ellipse whose semi-major axis $ON = n$ and semi-minor axis $OH =$ unity, and center O at distance $\sin \alpha$ from A.

The Locus of the Cusps C.—The rosette in Fig. 7 has three cusps, and in general it is evident that the number of cusps must be equal to p, the number of cycles of the pen, and that their angular intervals must be $360°/p$. For this reason we may confine ourselves to the cusp C that is first drawn after the pen has been set down at B and call it *the* cusp.

To find the phase of the pen when it is at the cusp, we observe that the pen must then be momentarily at rest, so that its rectilinear speed must be equal and opposite to its rotary velocity. As the latter is always at right angles to the radius through A, the harmonic motion can be so only when the pen crosses the X axis at L, so that one value of RL is the sine of the phase of the pen when at the cusp C and the other when it is at F, where the linear and rotary speeds are also equal but in the same direction. The rotary speed of the pen when at C is $ACd\theta = ALd\theta = n \sin \beta d\theta = \sin \beta d\beta$, its rectilinear speed at L is $d(LR)$ so that $LR = \int \sin \beta d\beta$ is equal to $\cos \beta$ or $\sin (90° \pm \beta)$ in absolute value. Of these two, $90° + \beta$ must be the phase for the cusp C, because as the rotary motion carries the pen anticlockwise, the rectilinear motion at L in Fig. 7 must then carry it in the opposite direction, that is, clockwise or downward, so that the phase must be greater than 90°. For the point F the phase must then be $90° - \beta$.

The phase of the harmonic motion at the cusp C being $90° + \beta$, that of the rotary motion must be $(90° + \beta)/n$. The angular position of C on the disk therefore varies directly as β, and as its linear distance from A is $n \sin \beta$, its locus must have an equation like $\rho = n \sin n\theta$, and when $n = 3$ as in Figs. 7 and 8, this is the equifoliated and non-cuspidal trifolium shown in dotted lines in Fig. 8, which is exactly like Fig. 5 but n times as large. The position of the axis of the first lobe is found by making $n \sin \beta$ a maximum, that is, $\beta = 90°$, so that (see Fig. 8 in which four rosettes are given with $\beta = 0°, 30°, 60°, 90°$) the phase of the pen $90° + \beta$ becomes $90° + 90° = 180°$, and the phase of the disk $180°/n$.

This last value directly locates the 0° phase of the disk, which may however be found also from any value of β, since it is equal to $(90° + \beta)/n$ as said before.

The arc AC of the cusp folium is equal to the elliptic arc HB on Fig. 7, and similarly to any corresponding elliptic arc on Fig. 8, that is to say, the distance of the cusp C from the center of the disk A as measured along the arc of the cusp folium is equal to that of the starting point B from the Y axis as measured along the ellipse. In the rectification of a polar curve the length of an arc is

$$s = \int (d\rho^2 + \rho^2 d\theta^2)^{1/2},$$

so that in the cusp folium where $\rho = n \sin n\theta$, we have

$$s = \int (n^2 \cos^2 n\theta \cdot n^2 d\theta^2 + n^2 \sin^2 n\theta \cdot d\theta^2)^{1/2}$$

$$= n \int_0^\beta \left(1 - \frac{n^2 - 1}{n^2} \sin^2 n\theta\right)^{1/2} n d\theta,$$

which is like

$$s = a \int_{\phi_0}^{\phi_1} (1 - e^2 \sin^2 \varphi)^{1/2} d\varphi,$$

the length of an arc of an ellipse with semi-major axis a, eccentricity e, and eccentric angle φ, as given by Byerly in his *Integral Calculus*, page 121, because in our ellipse $a = n$, $b = 1$, $e^2 = (a^2 - b^2)/a^2 = (n^2 - 1)/n^2$. Therefore the cusp folium arc AC equals the elliptic arc HB, and the perimeter of one lobe of the cusp folium is equal in length to the semiperimeter of the ellipse. This expression is true for values of $n > 1$, but when $n < 1$ it takes the form

$$s = \int (1 - (1 - n^2) \cos^2 n\theta)^{1/2} n d\theta$$

$$= b \int (1 - e^2 \cos^2 \varphi)^{1/2} d\varphi,$$

so that as a and b then exchange names and a cosine appears in the place of a sine, the elliptic arc is now reckoned from the end of the major instead of the minor axis, the zero point H however being always on the axis of Y.

The Locus of the Points of Contact F.—As the phase of the pen is $90° + \beta$ at the cusp C and $90° - \beta$ at the point of contact F, all that has been said about the former applies to the latter, provided we reverse the sign of β. The F points are thus seen to lie on the lobes of an equal equifoliated rosette with the equation $\rho = -n \sin n\theta$, these lobes being diametrically opposite to those of the cusp rosette. It has been omitted from Fig. 8.

The Tangent Method of Tracing the Rosette.—The mechanical method of tracing a rosette by having a pen move with a rectilinear simple harmonic motion of n cycles over a uniformly rotating disk, is mathematically a tangent method. If we imagine the two components of the pen's motion, the rectilinear over the

line ERG and the rotary about A, to act successively instead of simultaneously, the pen is first at L in Fig. 7 in its rectilinear motion, in phases $90° + \beta$ and $90° - \beta$, and is then carried to C and F by the rotary motion on a circle with center A and radius $n \sin \beta$. The length of the tangent to this circle is then zero. At any other phase θ the pen in its harmonic motion is, say, at B', which may be anywhere, but which we may place on B in order not to congest the

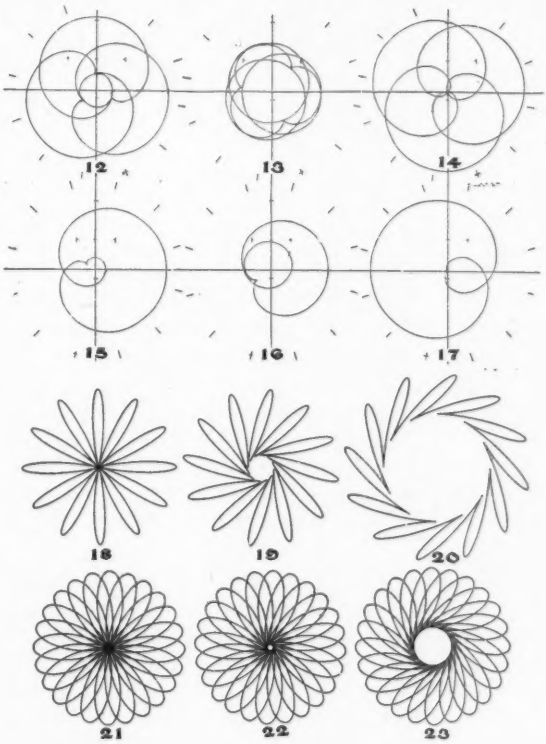


figure and which we accent in order to distinguish the two. While the rotary motion alone may be conceived first to carry the pen to D, the tangential harmonic motion then moves it along the tangent DP which is equal to $LB' = LR - B'R = \cos \beta - \sin \theta$, or rather $\sin \theta - \cos \beta$, because the tangent DP in the case illustrated is really negative.

On Fig. 7 these tangents have been drawn to almost every 30° of the disk. Owing to the small numerical value of LE in the instance presented, the tangents to the *tangent* circle, as we may call it, are positive only between F and C, between phases $90° - \beta$ and $90° + \beta$, which are here 60° and 120°, and negative for all other phases. The points C and F on the curve in phases $90° \pm \beta$ are

thus at minimum, and those in phases 90° and 270° at maximum distances from A. It is obvious, of course, that as the harmonic motion is n times as rapid as the circular, the tangent at any phase angle on the tangent circle runs to a point on the curve in n times that angle, so that 60° on the circle in Fig. 7 is joined to $3 \times 60° = 180°$ on the rosette, 330° on the circle to $3 \times 330° = 2 \times 360° + 270°$ on the curve, and so on.

The Equation of a Cuspidal Rosette.—The tangent DP in Fig. 7 ought to lead us to the equation of the rosette. The radius vector ρ or AP is seen to be such that

$$\overline{AP}^2 = \overline{AD}^2 + \overline{DP}^2$$

or

$$\rho^2 = a^2n^2 \sin^2 \beta + a^2 (\sin (n\theta + \alpha) - \cos \beta)^2$$

in which a is the amplitude of the simple harmonic motion of the pen ER, which we may take as our unit, n, β, α, have their usual meaning and any assumed values, and θ is the position angle of D from $+ X$, or the angle DAL. But it is the angle PAL that we need. Calling this ω and DAP δ, we have $\omega = \theta + \delta$, using the plus sign in general because when DP is negative, as it is in Fig. 7, it will reverse the sign of δ. Now δ is the angle whose tangent is DP/AD. Taking θ as the independent variable, we may find ρ and ω, but it does not seem possible to express the relation between ρ and ω in one equation without the help of θ.

The Three Elements of a Cuspidal Rosette, n, α, β.—There are three elements that determine the shape and position of a cuspidal rosette, not to mention its size which depends upon the amplitude of the harmonic motion that we take as our unit. The first element is $n = p/q$, the ratio of the cycles of the pen p to those of the disk q. While n might be incommensurable, only simple ratios of integral numbers are here considered. The second element is α, the initial phase in its harmonic motion at which the pen is set down on the disk. This may have any value from 0° to 360°. In this paper α is taken as less than 90°, greater values having been treated in the cardioid article. The third element is β, the eccentric angle of the point B on the ellipse at which the pen is set down on the disk.

Variation in the Elements.—The nature of a rosette obviously depends upon $n = p/q$ and β, since p determines the number of lobes and cusps and q the number of its convolutions about the center A, while β, that is, $n \sin \beta$, modifies its shape by determining its distance from the center. There is then nothing left for α to do but to fix the position of the curve. For if the pen is first started in phase 0°, and then in phase α, the advance of the pen on the disk will be α/n, and the whole rosette will be shifted that angular amount forward in a clockwise direction. Hence the disk reading for $+ X$ then becomes α/n instead of 0°, as it is in Fig. 8, where $\alpha = 0°$. In Fig. 7 therefore, where $\alpha = 20°$ and $n = 3$, the circle reading for $+ X$ is $\alpha/n = 6° 40'$. The four points of the circle, 0°, 90°, 180°, 270° may thus be properly marked and intermediate radii drawn for every 30° at pleasure. Then, as the circle reading for the axis of the first cusp lobe was found to be $180°/n$, its position angle from $+ X$ then becomes $(180° - \alpha)/n$.

A variation in α alone shifts the center O of the ellipse along the Y axis to the distance $\sin \alpha$. It has no effect on the nature of the curve, as has been said, so that the rosette in Fig. 7, in which $n = 3$ and $\beta = 30°$ but $\alpha = 20°$, is exactly equal in every respect except as to the axes of X and Y to the second rosette on Fig. 8 in which also $n = 3$ and $\beta = 30°$ but $\alpha = 0°$. Even the position is the same in regard to the circle reading, because this is $(90° + \beta)/n$ for the cusp, independent of α, as we saw before.

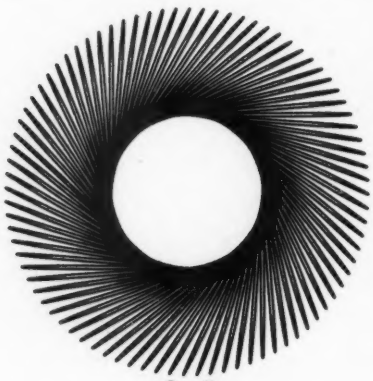
A change of α, as has been said, affects merely the position of the rosette, so that it is drawn sooner or later than it was at the first value of α. While the curve is being drawn there is then no reason why we should not be able to take any instantaneous position of the pen as an initial position. As this initial position must be at the assumed starting point B at the eccentric angle β of the starting ellipse mentioned before whose center is the distance $\sin \alpha$ from A in the direction of B from R (Fig. 7), we may take this assumed point B on the ellipse, move it together with the ellipse along the rosette to a point in another phase, swing the ellipse about this second position of B until its center is at the distance of the sine of this new phase from A on a line parallel to the direction of the harmonic motion of the pen at the moment, and then mark the center on the paper. It we do this for all points of the rosette, we shall find the locus of the ellipse center O to be an equifoliated non-cuspidal rosette exactly like Fig. 5 when $n = 3$, with its lobe axes of unit magnitude lying on those of the contact F folium. The reason is that $AO = \sin \alpha$, and the angular revolution of the ellipse about B must be uniform for equal phase intervals, so that $d\theta$ is constant when $d\alpha$ is, as we always consider it to be. This O-folium may readily be drawn for any value of n, if we place the pen at A when in phase 0°. No illustrations of the O-folia are here given, since they would be for all values of n, as much like the original rosette when $\beta = 0°$ or 180° as Fig. 5 is like Fig. 1, that is to say, the O-folia would be non-cuspidal and of unit magnitude.

A variation in β alone does not affect the nature of the rosette when the old value of β is added to or subtracted from 180° and 360°. For this reason it is most convenient to use values of β less than 90°, and to express greater values in the way indicated. For $180° + \beta$ the new curve is symmetrical to the β rosette with respect to the center of the disk A. For $180° - \beta$ and $360° - \beta$ it is symmetrical to the β curve with regard to the $0° - 180°$ and $90° - 270°$ diameters respectively of the circle reading.

But it is the variations in n that teach us most about rosettes. Accordingly Figs. 9–25 show some typical cases. For Figs. 9–17 the initial phase α has been taken as 52° for the sake of comparison. The starting point B and the center O of the starting ellipse and the extremities of its axes have everywhere been marked. The dozen radiating dashes indicate every 30° of the circle, the 0° being marked by a cross. This last applies also to Fig. 2, in which however $\alpha = 90°$ and $\beta = 0°$. In Figs. 9, 10, 11, $n = 4/3$ and $\beta = 30°, 90°, 180°$, while in Figs. 12, 13, 14, $n = 3/4$ with the same values of β. In Figs. 15, 16, 17, $n = 1/2$ and $\beta = 30°, 60°, 180°$. When the denominator q in n is large, the

number of convolutions of the curve are uninterestingly numerous in proportion.

As α determines only the position of a curve, and this is generally of no consequence, the most convenient value of α to use is 90° when $\sin \alpha$ is a maximum and the pen is at one extremity of its harmonic path. Then when n is large and β small, the starting point is practically on the X axis. Thus Figs. 18, 19, 20 show the ratio $n = 12$ and $\beta = 0°, 3°, 19°$, respectively. In Figs. 21, 22, 23, $n = 24/5$ and $\beta = 0°, 1° 4', 5° 42'$, and in Figs. 24 and 25 $n = 84$, $\beta = 0° 38'$ and $\pm 0° 38'$. This ratio of $n = 84$ is the largest the author's machine is at present able to produce. The machine was described in the *Scientific American Supplement* of February 9 and 16, 1918. Fig. 25 is in a sense merely the double



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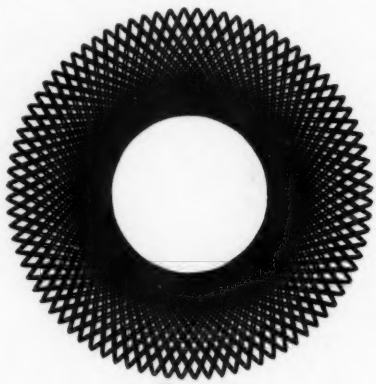

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of Fig. 24. It was drawn by first tracing Fig. 24 with the B point to the right of the center, and then by starting the pen an equal distance to the left. The cuspidal nature of some of the rosettes presented is not conspicuous, especially when β is very small, on account of the closeness and apparent superpositions of the paths of the pen when near the cusp. While other and more beautiful rosettes could have been drawn, had the restriction as to cusps been waived, they would have been foreign to the present study.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

DISCUSSIONS.

What are the essential ideas which should be retained permanently by a student of the calculus, after the details and the special problems have faded from his mind? That the calculus makes a valuable contribution to the general culture of a student who has no need to make practical use of it, will not be

questioned by any of us who teach it. Too often, however, we fall into a contented routine of meeting our classes with self-satisfied assurance, assigning the regular lists of problems, conscientiously explaining to the students how to surmount their difficulties with Problem 7 or Formula 42, and leaving our teaching dead with lack of emphasis and discrimination. Not seldom we fail to point out the beauties of the forest because we ourselves see only the trees. To bring us back to fundamentals such an article as Professor Rietz's discussion below must be welcomed by every earnest minded teacher of the subject.

The second discussion, by Professor Bennett, points out the disadvantages attendant on each of the usual conventions for attaching an algebraic sign to the distance between two points or between a point and a line, in analytic geometry. He concludes that the plan most commonly found in our text-books (which he calls the transcendental determination) presents more serious difficulties than either of the others. We are glad to endorse his request for "a reply in defense of the accepted methods."

As the last discussion, we present an interesting little study by Mr. B. H. Brown on a subject which has probably not hitherto received so detailed a mathematical treatment. The close agreement between theory and practice in the probabilities of various occurrences connected with "crap-shooting" is surprising. Some of the questions discussed may furnish suitable material for class problems, as for example the verification of the figures in the columns marked "Theoretical" at the beginning of the article. Devotees of the game will scarcely find in the article arguments to support its availability as a rapid road to riches.

I. On the Teaching of the First Course in Calculus.[1]

By H. L. Rietz, University of Iowa.

In his presidential address[2] read before the American Mathematical Society, Osgood expressed the view that "the calculus is the greatest aid we have to the appreciation of physical truth in the broadest sense of the word." This fact should serve to keep before teachers of the first course in calculus the following important question: Can we teach this course so that our students will feel more keenly than at present the use of the calculus in the appreciation of physical truth?

One needs only compare the text-books in calculus at present with those in use thirty years ago to be convinced that much progress has been made along this line. The first course in calculus has been enriched through a broad range of problems from everyday experience and from physical science. Moreover, logical flaws centering around the conceptions of the infinitesimal and differential have been so largely removed that the mental satisfaction derived from a close contact with physical problems has been much increased. This

[1] Read before the Iowa Section of the Mathematical Association of America, April 26, 1919.
[2] *Bulletin of the American Mathematical Society*, Vol. 13, p. 449.

does not mean that time is given in the first course to rigorous demonstrations, but that time is saved through removing mysticism by a clear statement of definitions and assumptions. It goes without saying that further improvement in the problem work of the course will lead towards the attainment of that object of the calculus expressed in the above quotation, and that all teachers of the subject should devote much thought to new problems for the course. In the present paper we shall say very little about the character of problems, as it is our purpose to emphasize mainly another but related phase of the teaching of the calculus.

It has been a matter of interest to me to test the results of my own teaching of the first course in calculus and those of other teachers by asking students some years out of college, say ten years out of college, about the mental picture they retain of the nature and uses of the calculus. Naturally great variations occur in the picture, depending upon the ability and the life work of the man in question. It has been particularly interesting to me to consider on the one hand, the case of the physicist, the chemist, the actuary, and the engineer; and on the other, that of the man whose life work calls mostly for qualitative rather than quantitative thinking. Some of those who are making no professional or occupational use of the methods of the calculus seem to retain only a general and rather vague notion of certain formulas and transformations applied to the finding of areas and volumes. Indeed, with respect to most of the men who have little quantitative thinking to do the formulas are forgotten. But it has been a source of great satisfaction to find that many of these men long out of college have a picture of the method of the calculus that shows something of how it has entered into their thinking. They still picture the differential calculus as a method which gives us a precise conception of the rate of change of functions and quantities in the phenomenal world. Some have told me that in the integral calculus, they learned to add an infinitely large number of infinitely small quantities. One of my students who took calculus several years ago and who has to make no direct use of the calculus told me recently he could never forget this as fundamental in the calculus and that he appreciates to some extent the importance of this problem because most of the things about us seem to be made up of an infinitely large number of infinitely small parts. Although this conception of adding an infinitely large number of infinitely small parts may savor of an archaic view of the infinitesimal and involve a certain form of mysticism, still I would far rather find that the student has retained such a picture of the calculus than a picture of forgotten formulas.

For our consideration as teachers of the first course in calculus let me submit a brief description of what seems to me should be the minimum of detail in the mental picture retained by the average student ten or more years after studying a first course in the calculus. It seems to me the picture should include at least the following:

(1) That the determination of the rate of change of any quantity is a problem of the differential calculus and that this problem is of almost universal interest in relation to quantitative phenomena.

(2) That by the methods of the differential calculus approximate rates of change of quantities are replaced by exact formulas.

(3) That any limited physical quantity can be represented, with an approximation as close as you please, as a sum of simple and similar component elements, and that by the methods of the integral calculus such approximate sums are replaced by exact formulas.

(4) That physical laws are very generally formulated as differential equations.

The man who has formed a picture, with these notions as the main lines, to aid in the appreciation of the physical world may well be said to have formed "the calculus habit" in his thinking. It is our problem to teach the first course in calculus so as to establish this calculus habit of thought in our students. The important question in this connection is: By what means can we more successfully accomplish this end in our teaching? We would no doubt agree that each of the four points named above should be illustrated many times by concrete problems representing situations in everyday life and in the phenomenal world. We are no doubt doing much at the present time in our teaching of the first course in calculus that tends to carry out the purpose which I have in mind. Thus, we are all accustomed to impressing upon our students the necessity of looking upon a rate of change with respect to time as the limiting value of an average rate of change. We thus replace the approximate rate by an exact conception. We are also no doubt carrying our problems on areas, volumes, masses, moments of inertia, energy, and work back to the central point of looking upon the quantities as limits of the sum of simple and similar component elements. And yet, with all the beautiful and interesting problems that are given and will be given in the first course in the calculus, some of us have a strong feeling that the appreciation of the subject could be increased if more emphasis were placed on the fact that the solution of these problems may well be regarded as interesting details of a picture whose main lines are in a few fundamentals such as the four points mentioned above. The habits of the average student and perhaps even those of the exceptional student are formed largely through repetition. If we would have our students form "the calculus habit" of thinking or would make the calculus such an aid for them as it should be in their appreciation of physical truth, we must recall the main features of our picture whenever we fit into it the various applications to problems of physics, chemistry, geometry, engineering, and everyday life. Otherwise the calculus habit in thinking is not likely to be formed.

In thus contemplating the picture which we would have our students form of the calculus as a fundamental aid in the appreciation of physical truth, it is not to be inferred that the technique or formal side of the calculus may be neglected as unessential. Knowledge of the elements of the calculus is acquired largely through formal work. It is only after attaining considerable proficiency on the algebraic side of the calculus that one is ready to form the picture that should be retained. In this connection it is sometimes maintained that a knowledge of the derivatives and integrals of a few simple functions is adequate prepara-

tion in the calculus for an engineer. This position is not without a certain element of truth. It is possible to state the integrals necessary to follow an elementary treatment of physics and mechanics in brief space, but a well-grounded knowledge of the subject that will abide with the student is not to be obtained by confining the algebraic work within very narrow limits. It seems to me that with respect to technique neither the extreme view of algebraic formalism that was in vogue twenty-five years ago in the first course in calculus based on certain English textbooks very generally in use in this country nor the practical lack of formalism advocated by those who would popularize the calculus, makes a satisfactory contribution to our picture of the first course. Surely without approximately the amount of technique given in our recent textbooks we should give totally inadequate preparation for the serious study of mechanics, electricity, thermodynamics, rigid dynamics, hydrodynamics, the theory of elasticity, or for the reading of much of the recent literature of physics and chemistry. Indeed, a second course in calculus has been so strongly urged by certain prominent electrical and mechanical engineers that a course has been introduced into the junior year in certain schools to meet the demand. It goes without saying that technique is indispensable to the student who is to become a mathematician and it is clear that we should take due account of this class of men in looking to the future of education along mathematical lines. It seems to me that some of our recent textbooks in the elements of calculus exhibit rare judgment on the part of the authors in their position with respect to the amount of formal work to be done in the first course in calculus.

In conclusion, let me submit that we can make the first course in calculus a more important aid in the appreciation of physical truth by aiming more directly at establishing in the minds of our students what I have tried to indicate as the calculus habit of thinking about phenomena and that this result is most likely to be obtained by creating for the student a mental picture with only a few main lines.

II. The Sign of the Distance in Analytical Geometry.

By Albert A. Bennett, University of Texas.

The pupil in a course in analytical geometry comes very early across two formulas concerning distances. These yield the following schematic equations:

$$[\text{distance from } (x', y') \text{ to } (x, y)]^2 = (x' - x)^2 + (y' - y)^2,$$

$$[\text{distance from } (x', y') \text{ to } ax + by + c = 0]^2 = \frac{(ax' + by' + c)^2}{a^2 + b^2}, \qquad (a^2 + b^2 \neq 0).$$

No discussion of the derivation of these formulas need be made here. The question of present interest is what is to be done in determining the algebraic sign of the distance itself, that is the square root of each of the above expressions. Now the pupil has become accustomed in dealing with square roots to employ conventions of three types with respect to the determination of signs:

1. Numerical determination, where the positive sign only is used.

2. Transcendental determination, where both positive and negative signs occur, but the choice in any given instance is determinate.

3. Algebraic determination, where the sign is indeterminate, and both possibilities as to sign are retained throughout.

The numerical determination is that regularly occurring in elementary arithmetical and geometrical problems, and in many algebraic examples. The transcendental determination is used in the basic conventions of analytical geometry and trigonometry, and the algebraic convention is the rule in algebra except in certain numerical problems. The existence of these distinct conventions is only vaguely present to the pupil's mind but is a frequent source of misconception and error, and merely adds to the student's conviction of the arbitrary and blind character of mathematical notations. But the different points of view are not simultaneously consistent and each has its place, a fact which should at least be appreciated by the teacher.

Each of these three possibilities as applying to the sign of the distance will be treated in turn.

1. **Numerical Determination.** In elementary arithmetic, subtraction is taught, but negative numbers are not usually admitted. A quadratic equation as such does not arise, but the notion of square root is frequently in evidence. One thinks in every case of but a single square root, which is positive. In geometry a construction is given for the mean proportional, and usually mention is not made of two mean proportionals of opposite sign but of one only. Problems in mensuration such as: define the altitude of an equilateral triangle with an integral side, the radius of a circle of given area, etc., lead to square roots, but always to positive quantities. And so the fact that lengths and distances in everyday life are never negative should not surprise the mathematician. The distance from New York to Chicago is equal to the distance from Chicago to New York, the shortest distance from Denver to the Canadian border is equal to the shortest distance from the Canadian border to Denver, etc.

Thus when the student is given an elementary locus problem it might appear natural to regard distances as inherently positive. If a straight line be drawn in a plane, where no reference is made to coördinates, and it be desired to find the locus of points one foot from the line, it seems obvious that the locus will be two lines, not one only. If two points be given, the length of the segment joining them should be the distance between the two points, and the length is inherently positive as usually treated. The locus of points in a plane equally distant from two given intersecting straight lines, each line being viewed as extending indefinitely in both directions, is of course the pair of bisectors of the angles, a pair of perpendicular straight lines. It seems hardly necessary to reinvestigate the question as to whether the center of the inscribed circle of the triangle is equally distant from the sides, and the centers of the escribed circles equidistant from one side and the extensions of the other two sides, or whether the center of the circumscribed circle is equidistant from the vertices. To check up such facts may

serve as good pedagogical material but that a convention should retain these relations as still true can occasion no comment.

The numerical determination is, however, subject to the following serious drawback in the elementary treatment of analytical geometry: It requires every algebraic solution to be supplemented by a more or less tedious examination of extraneous solutions for which no adequate preparation is usually made. A few examples will be cited which by their simplicity serve to illustrate sharply the awkwardness inherent in this convention.

1. Find the locus of a point such that the sum of its distances from the rectangular axes is equal to unity.

The locus is a square with vertices (1, 0), (0, 1), (− 1, 0), (0, − 1). Here the algebraic work leads to four lines of which only the segments forming the sides of the square satisfy the conditions. The extensions of the sides are extraneous.

2. Find the locus of a point such that the sum of its distances to the two rectangular axes is equal to the distance to the line whose equation is $4x + 3y - 5 = 0$.

The locus is an irregular quadrilateral with vertices on the axes. Only the points of the segments between adjacent vertices satisfy the problem.

3. Find the locus of a point such that the sum of its distances to the two axes and its distance to the line $4x + 3y - 5 = 0$, is unity.

The locus consists of a single point, the origin; all other parts of eight lines are extraneous.

4. Find the locus of a point such that the sum of its distances to the two axes and its distance to the line $4x + 3y - 5 = 0$, is two.

The locus consists of an irregular convex hexagon, whose opposite sides are not equal in length but are parallel.

5. Find the locus of a point such that its distance from (0, 1) exceeds by unity its distance from the X-axis.

The locus consists of one entire parabola tangent to the X-axis at the origin together with the negative half of the Y-axis.

6. Find the locus of a point such that its distance from (0, 1) exceeds by two its distance from the X-axis.

The locus consists of parts only of two parabolas, both having (0, 1) as focus, but with different latera recta, the curves having a common chord on the X-axis and being joined there, but not crossing each other, the portions nearer the focus being extraneous.

7. Find the locus of a point such that the sum of its distances from (1, 0) and (− 1, 0) is two.

The locus consists of the segment of the X-axis joining these two points. For the remainder of the X-axis the condition is that the difference of the distances be two, and if the subtrahend and minuend be specified, that one of the portions which is intended is given.

8. Find the locus of a point such that the sum of its distances from the three sides of an equilateral triangle is a given constant.

When the given constant is less than the altitude of the triangle, the locus contains no points.

When the constant is equal to the altitude of the triangle, the locus contains every point within and on, but no point exterior to, the triangle. The locus is in this case not a curve but a portion of the plane.

When the given constant is greater than the altitude of the triangle, the locus consists of a hexagon surrounding the triangle.

In conclusion, the numerical determination while entirely natural and self-consistent, appears to introduce unnecessary complications and non-algebraic investigations not particularly adapted to an elementary course in analytical geometry.

2. **Transcendental Determination.** Analytical geometry became possible by a conscious selection of special conventions facilitating the treatment of negative quantities. Having selected an axis for X's, no algebraic study could determine which direction would be the positive and which the negative. By a non-algebraic and essentially transcendental step one side is adopted as the positive half and the other as the negative. Even when the X-axis is agreed upon, the convention for Y's remains arbitrary. The question of sense is not settled by symmetry. The choice adopted makes the equations for rotation of axes appear in the unsymmetrical form

$$x' = \quad x \cos\theta + y \sin\theta,$$

$$y' = -\, x \sin\theta + y \cos\theta,$$

rather than in the other possible but equally unsymmetrical form. Seeing what advantages are secured by this arbitrary choice, it is but natural that similar advantages be expected from an arbitrary convention with respect to distances throughout. The entire theory of a coördinate system is to regard given distances as associated with opposite signs when measured in opposite directions, and more than this, to select one direction in each case as positive. Now a line separates the rest of the plane into two parts. If a point be on one side, it cannot move continuously to the other side without passing across the line. What is more obvious than to assign to the distance an algebraic sign, such that one side is reserved for positive distances and the other for negative, these being separated by the line itself whose points are at zero distance? This convention is made in nearly all, if not all, standard American textbooks in analytical geometry. The choice used in the various particular cases is easily suggested by the fact that the origin is always present as a datum point for any line not containing it, and for lines through the origin, a convention consistent with that for the axes is desirable. The convention as to distance between points is usually the numerical determination already mentioned, since otherwise the points of a circle might not all be equally distant from the center.

The transcendental conventions are open to very serious objections, however. For example, the distance of a point from a straight line even if it be selected for conve ence as an axis of reference is not independent of the rotation of axes,

despite the explicit statements in most of the books concerning the invariance of distance, since the sign changes in certain cases. But consider some selected examples:

1. Find the locus of a point whose distance from the origin is equal to its distance from the Y-axis.

The locus consists of the origin and the positive half only of the axis of X's. Points on the negative half of the X-axis have the opposite sign.

2. Find the locus of a point the sum of whose distances from the two axes is equal to its distance from the origin.

The locus consists of the two positive half axes with the origin.

3. Find the locus of a point at a given constant distance from a straight line.

The result of this problem is somewhat involved. If the equation of the line as well as the sign of the distance be known, there is but one answer: a single line parallel to the given line and on the determinate side of it. If the equation of the line is not given, one of two possible lines is clearly the answer, but which one? The transcendental theory prohibits two answers, and requires that one be selected in a prescribed manner, but does not furnish the data. If a student does as he is told and arbitrarily selects the coördinate system in some convenient way, he will obtain a unique and supposedly correct answer. But two different students may each assert that his own solution is unique and correct, while these may correspond to opposite determinations of sign.

4. Find the locus of a point such that the distance between the point and a given line is a given constant.

This is merely a catch question and has no answer, because the distance is either from the line to the point or from the point to the line and these are distinct. There is no single distance, unless it be zero, between the point and the line.

5. Find the locus of points equidistant from two given intersecting lines.

This problem is impossible unless the position of the origin be known, and should the origin be on one of the lines, the position of the axes must also be given. These data known, the answer is a single line.

6. Find the center of the inscribed circle of a given triangle.

This is but occasionally a point equidistant from the three sides. This being a point determined by the figure and not by the coördinate system, all special transcendental conventions are useless. The point equidistant from the three sides by the transcendental convention, may be tested algebraically or graphically and has roughly only one chance out of four of being the center of the inscribed circle.

7. Find the locus of points equidistant from two given parallel lines.

The locus consists of no points, unless the origin should fall within the two lines, or in certain cases upon one of the lines.

8. Given AB perpendicular to BC, find the distance from A to BC.

This problem is impossible unless the position of the origin be known. The distance may or may not be equal to the positive length AB.

9. Calling the locus of points one unit distant from a given line, the line one

unit distant from the given line, find the envelope of the lines one unit distant from the lines through the origin.

The envelope is not a circle, but half a circle.

10. Given a fixed point, (x_0, y_0), other than the origin. Any line through (x_0, y_0) will have an equation of the form, $(x - x_0)\cos\theta + (y - y_0)\sin\theta = 0$. Determine the character of the distance from any point, (x', y'), to the line, $(x - x_0)\cos\theta + (y - y_0)\sin\theta = 0$, as θ varies.

The distance is a periodic function of θ. If (x', y') is not collinear with the origin and (x_0, y_0), the distance has a finite discontinuity as $\tan\theta$ passes through the value $-x_0/y_0$. When (x', y') is on the line joining (x_0, y_0) and the origin, the distance is constantly positive or constantly negative, depending upon the order of the points $(0, 0)$, (x', y'), (x_0, y_0), on this common line.

The transcendental definition has the great advantage of emphasizing the fact that the sign of the distance from a line may be made of use in determining whether or not two points given by their coördinates are on the same or opposite sides of a given line, but in so far as it specifically determines a positive or negative sign, there seems to be no advantage derived. One might protest in behalf of the fundamental conventions of analytical geometry with regard to coördinates. A reply is, however, obvious. The coördinates may be held to involve more than distances, they involve directions and this distinction is regularly observed at the start. The simpler geometric notions of distance are independent of the position of coördinate axes, and only suffer by special conventions attempting to relate them.

3. **Algebraic Determination.** As is usual in algebra one can regard distances as inherently either plus or minus as far as algebraic distinctions are concerned. It is possible to explain and emphasize the fact that in any particular example a fixed sign may be given to the expression, which sign is selected being clearly immaterial, and so the fact as to whether two points are on the same or opposite sides is readily determined. Mention may be made with regard to the convenient character of the origin as a reference point when it happens to be given. At the same time any complete answer to an exercise must take into consideration the possibility that the student has a free choice in sign and the solutions will be as numerous as the possible combinations of signs. This is just what is done throughout in algebra. For example, if $x^2 - 3x + 2 = 0$, be an equation proposed and it is desired to find the solution x intended by the proposer, the problem is not possible algebraically. One can only say that x must have been either 1 or 2, and more cannot be determined except possibly by other methods. This does not mean that the proposer had necessarily these two solutions in mind. He may have intended one or the other or both, but which possibility cannot be found by algebra alone if at all. Thus the locus of points at one unit's distance from a given line, may be viewed as a single line on one side, or one on the other side or as both lines. The only legitimate answer appears to be a pair of lines, with the understanding that either one alone might be the answer intended by special conventions. The algebraical determination has all the completeness of

the numerical determination, and all the flexibility of the transcendental. It has also its own objections, however. The algebraical determination always gives complete algebraic curves instead of portions as sometimes given by each of the other conventions. Its disadvantage is that plus a distance and minus a distance are not distinguished. Let us examine some instances.

1. Find the locus of a point such that the sum of its distances from two given points is a constant.

The solution is either an ellipse, an hyperbola, or a straight line, according to the relation of the constant to the distance between the two given points. By the numerical definition, the ellipse and part of the line correspond to sums, and the hyperbola and the rest of the line to differences, the two branches of the hyperbola, and the two rays of the line corresponding to the two choices of subtrahend and minuend.

2. Find the locus of a point such that the sum of its distances from the rectangular axes is equal to unity.

The locus is four complete straight lines, viz.,

$$x + y = 1, \quad x - y = 1, \quad -x + y = 1, \quad -x - y = 1,$$

parts of the lines corresponding in the numerical definition to distances.

3. Find the center of the inscribed circle of a given triangle.

The three lines of the triangle have each two sides which are half planes. The two sides may be at once distinguished algebraically, if the opposite vertex be in each case compared with any point under consideration. Thus of the four points obtained as equidistant from the lines, that one which is simultaneously on the same side of each line as the opposite vertex is the center of the inscribed circle. If an origin be given not on any of the lines, or any other point be convenient, this point might be used as reference for all three lines if preferred.

Conclusion. No teacher or good student is badly misled by the transcendental conventions as to distances in the analytical geometry textbooks. He uses the convention when convenient and rejects it otherwise. The definitions there given are, however, baffling to the poor student, and either examples must be skilfully selected in the textbook or by the teacher to avoid trouble, or else the conventions which have been so carefully drilled into the memory of the unreasoning must be denied at just those moments when the pupil feels most helpless. It seems time to reject the transcendental concepts which are neither simple nor tenable, in behalf of either the obvious but difficult numerical determination or the simple but perhaps over-inclusive algebraic determination. The author realizes, however, that this cannot be a new question and it is one which must have often been examined by teacher and by textbook writer. A reply in defense of the accepted methods might prove of general interest.

April, 1919.

III. Probabilities in the Game of "Shooting Craps."

By Bancroft H. Brown, Harvard University.

Webster's *International Dictionary* gives the following explanation: "The caster throws or 'shoots' the dice, and wins if the throw is 7 or 11 (called a *nick* or *natural*), but loses if it is 2, 3, or 12 (called a *crap*). If 4, 5, 6, 8, 9, or 10 is thrown it becomes the caster's 'point,' and the caster continues to throw until he wins, by again throwing his point, or loses, by throwing 7. The odds are 251 to 244 against the caster."

This probability and the incidence of various points, craps, and naturals have been tested in a series of 9,900 games. Theoretical and actual incidences follow:

		Theoretical.		Actual.	
		Won.	Lost.	Won.	Lost.
Naturals	7	1,650		1,635	
	11	550		553	
Craps	2		275		259
	3		550		508
	12		275		293
Points	4	275	550	267	565
	5	440	660	481	662
	6	625	750	567	787
	8	625	750	620	738
	9	440	660	451	685
	10	275	550	297	532
Totals		4,880	5,020	4,871	5,029

In the further discussion of this game, use is made of the following theorem, which is believed to be new:

Theorem: *In any series of games where the probability of winning is constantly p, the average number of games won, up to and including the first lost game, is the reciprocal of the probability of losing.*

For the average number, A, is equal to the limit of the sum of the various probabilities that the first $(n-1)$ games are won and the nth lost multiplied by n, where n increases without limit, that is:

$$A = 1\cdot q + 2qp + 3qp^2 + \cdots + nqp^{n-1} + \cdots \text{ (where } q \text{ is the probability of losing)}$$

$$= q\cdot \sum_0^\infty (n+1)p^n, \qquad 0 < p < 1,$$

$$= q\cdot \left[\sum_0^\infty p^n\right]^2 = q\cdot\left[\frac{1}{1-p}\right]^2 = \frac{1}{q}.$$

We employ this theorem to determine the average number of rolls needed to decide a game. For example, if "6" is a point, the point is won or lost as "6" or "7" is thrown, and since of the 36 ways in which 2 dice may fall, 5 ways give "6," and 6 give "7," the probability of deciding the point on the next cast is 11/36. Hence 36/11 throws, on the average, are required to decide the point "6." Exactly the same reasoning holds if "8" is the point. For "5" or "9,"

A is 18/5; for "4" or "10," A is 4. The average of these, allowing for their relative incidence is $3\frac{31}{55}$; and the average number of rolls per game (including those games won or lost by naturals or craps) can now be shown to be $3\frac{62}{165}$. In general, we may say that in 557 rolls of the dice: 55 are naturals or craps, 110 make points, and 392 complete these points.

This result may be shown independently of the theorem quoted. Let x equal the average number of rolls which are needed to complete a point *inclusive* of the first roll. Then in N games, where N is very large:

$$\frac{N}{3}+\frac{2Nx}{3}=N', \qquad \text{where } N' \text{ is the total number of rolls,}$$

$$=6\cdot s, \qquad \text{where } s \text{ is the total number of "7's" rolled,}$$

$$=6\cdot\left(\frac{N}{6}+\frac{N}{9}+\frac{2N}{15}+\frac{5N}{33}\right)$$

whence x equals $4\frac{31}{55}$, and the conclusion follows as above.

According to best Army traditions, the caster retains the dice, and has the privilege of naming the size of the next bet as long as he wins, and also if he throws a crap; giving the dice and the privilege to his opponent only if he loses a point. The probability that he holds the dice is obviously 244/495 augmented by the probability of throwing a crap (4 chances in 36), that is, 299/495. The average number of games which he will roll is then 495/196 or 2.5255, of which he will win 1.2449 games, and lose 1.2806 games.

De Morgan[1] indicated the incidence of unusual runs of luck in several series of 2048 trials where the probability p was 1/2. In this question of holding the dice we have a probability of 299/495, practically 3/5. The first 2,471 games in the test mentioned above comprise exactly 1,000 turns, and an integral approximation of the theoretical together with the actual incidence is tabulated below:

	Theoretical.	Actual.
Total number of games	2,526	2,471
Total number of turns	1,000	1,000
Turns ending with 1 game	396	423
" " " 2 games	239	210
" " " 3 "	145	142
" " " 4 "	87	93
" " " 5 "	53	70
" " " 6 "	32	23
" " " 7 "	19	14
" " " 8 "	12	9
" " " 9 "	7	3
" " " 10 "	4	6
" " " 11 "	2	1
" " " 12 "	2	4
" " " 13 "	1	2
" " " 14 " (or more)	1	0
	1,000	1,000

CAMP DEVENS, MASS., May 5, 1919.

[1] Augustus de Morgan, *A Budget of Paradoxes*, 2d ed., 1915, Vol. 1, pp. 281–3.

RECENT PUBLICATIONS.

REVIEWS.

Elementary Mathematics for Field Artillery.

Elementary Mathematics for Field Artillery. By Lester R. Ford. (Prepared and published by direction of the Chief of Field Artillery.) Field Artillery Central Officers' Training School, Camp Zachary Taylor, Kentucky (Instruction Memo. No. 20.). Louisville, Ky., Courier Journal Press, 1919. 8vo. 72 pp.

Educators have recently become critical. Before deciding how a subject ought to be taught, it is now customary to question whether it ought to be taught at all, and to admit that the aim to be achieved may have an influence upon the methods to be employed. Elementary mathematics has been under fire in many quarters, and while the word "mathematics" has long ceased to be used as a plural noun efforts are but now being exerted in numerous directions to impress upon the pupil the unity of the science, particularly in coördinated freshman courses.

The war presented an exceptional although fleeting opportunity to emphasize the fundamental necessity of mathematics in the technical equipment of the individual. This was most conspicuous perhaps in the case of artillery and naval units. The student officer was required to learn and use many mathematical ideas that he had never acquired, or had frequently long forgotten. The result at least in many of the training camps was rather pitiable. The instructors themselves usually knew the formulas, but not the reasoning, and blind confusion was the rule rather than the exception among the students. In the S. A. T. C. program, the situation was nearly as bad. Even had classes not been interrupted, the students would never have acquired many of the essentials, for two reasons: much of the time was spent with unnecessary details, and again the instructors were not prepared in many cases to give what was needed. They did not know what concepts were employed, what problems were to be faced, and frequently what were the mathematical facts of the subjects even were these named. Such terms as mil, grad, deflection, azimuth, dispersion diagram, probable error, center of impact, nomogram, or alignment chart, meant little or nothing to many college instructors in mathematics, while most college graduates could not approximate a square root by the algorithm, determine visibility on a map with the contour lines given, interpolate with second differences, and the like. But all of these things had to be taught, not to a few only but to many. Not having been given even in a college course, these topics had to be learned from men of action rather than teachers, from soldiers who had never been concerned with mathematical theory.

It was in the face of an emergency of this sort that the present pamphlet

was prepared and issued, and although the military urgency has passed there is much suggestiveness in this little text-book, for the teacher of collegiate mathematics. The reviewer has learned that "about 15,000 students took the course in the three months prior to the signing of the armistice. The staff of instructors, recruited chiefly from the candidates who had mathematical training, was ordinarily well over a hundred, and at one time numbered as many as one hundred sixty-nine."

This pamphlet comprises sections entitled "arithmetic," "algebra," "geometry," "trigonometry," "approximate methods," "coördinates," "aids to calculation," and "probability," and has also an appendix of tables.

In the section on arithmetic the most interesting features emphasized are the conversion of metric into English units and *vice versa*, various units of angular measure, mil, grad, degree, the process of extracting the square root, and interpolation. The algebra and geometry sections cover customary topics; the geometry section is particularly rich in theorems, but many of the proofs are omitted. Under trigonometry, seven pages altogether, the sine, cosine, tangent, and cotangent are mentioned, but not the secant or cosecant. The law of sines and the law of cosines are derived and applied, but triangles with three sides given are not discussed and the law of tangents is not mentioned. Under "aids to calculation," logarithms, the slide rule, and a nomographic logarithmic table are explained. Under the topic of probability, the probability curve and the dispersion diagram are illustrated and applied. The tables are to four decimal places, and comprise (I) square roots of numbers, (II) logarithms of numbers, (III) natural trigonometric tables—mils, (IV) logarithmic trigonometric tables—mils, (V) natural and logarithmic tables—degrees.

This memorandum as an official publication contains of course neither preface nor introduction, but was obviously written with a single purpose in view—to serve as a text in teaching the field artillery officer the mathematics he must know. No topic that is necessary is omitted on account of its difficulty, nor is any topic of clearly minor importance for this object introduced because of its symmetry or historical interest. Nearly every problem mentioned in the exercises bears on its face the answer to the common query, "But why must I study this?" The work is written for adults, and is so condensed as to serve well for a reference book and source of problems but scarcely as a guide for self instruction of the mentally deficient. From a logical point of view, the book is surprisingly satisfactory. The definitions are clear and cover the terms used, the arrangement is progressive, where proofs are omitted there is no attempt to conceal the fact, processes are always explained in general terms, as well as applied to numerical examples, emphasis is placed on the operations used rather than on formulas—in short the book is written by a mathematician. In his experience in teaching from American text books designed for standard first year college courses, the reviewer has seldom found a treatment with so few logically objectionable elements; of the numerous mathematical memoranda he has seen, used in the army training schools, this is the first which shows ease and clarity of treatment.

A few of the exercises may be cited:

1. Near the muzzle of the three-inch gun the rifling makes one turn per 25 calibers. If the projectile leaves the muzzle with a velocity of 1,700 feet per second, how many rotations about its axis does it make per second? Ans. 272.

2. The total length of the French 155 mm. cannon is 2.332 m. Express this in calibers. Ans. 15.05.

3. In rising above the earth's surface the temperature falls about 6° C. for each kilometer of altitude up to eleven kilometers and thereafter remains practically stationary. What increase in elevation (to the nearest ten feet) will lower the temperature 1° F.? Ans. 300 ft.

4. For a range of 3,000 yards the 3-inch gun is elevated at an angle of 5° 5.3′ above the line joining gun and target. When fired the projectile actually leaves the gun at an angle of 92.3 mils about this line. How much is the former angle increased by the "jump" of the gun as it is fired? Ans. 1.8 mils.

5. A scout measures the angle between a line to an object C and the straight road along which he is passing. He proceeds along the road until, 1,300 yards farther, a line to C makes with the road an angle twice as great. How far is he then from C? Ans. 1,300 yds.

Note. This is the method used by sailors in "doubling the angle on the bow."

6. A reconnaissance officer has an accurate map of the surrounding region but does not know his exact position on the map. He sees two objects A and B, which are shown on the map. He measures the angle between the two points and draws lines through A' and B', the map positions of the two points, to meet at the angle found at some point C' on the map. Show that the true position is on the circle drawn through A', B' and C'. If he recognizes a third point whose map position is known he can get a second circle on which his position lies. His true position is at the intersection of these circles.

Note. This is the method of "Italian Resection" used to locate a position with a plane table.

7. In maps based on English units 1 in. to the mile, 3 in. to the mile, 6 in. to the mile, and 12 in. to the mile, are commonly used scales. Find the representative fractions of these maps? Ans. (*a*) 1/63360, (*b*) 1/21120, (*c*) 1/10560, (*d*) 1/5280.

8. Find the area of a shrapnel pattern 100 yds. long and 35 yds. wide. Ans. 2,749 sq. yds.

9. Find the danger space for a horse 15 hands high when the trajectory makes an angle of 252 mils with the ground. Ans. 20 ft.

10. An observer 2,000 yds. from the battery and 3,540 yards from the target finds the angle between the battery and the target to be 114 degrees. Find the range from the battery to the target. Ans. 4,721 yds.

11. Find the range and deflection components of a wind of ten miles per hour blowing from the southwest if the target is due east of the battery. Ans. range component = deflection component = 7.1 mi. per hr.

12. How far is the B. C. station from the right gun if the wheel of the gun carriage, which is 56 in. in diameter, subtends an angle of 4 mils? Ans. 390 yds.

13. Of ten shots fired at a range of 2,500 yds. eight are to the right of the target and two to the left of it. What change should be made in the deflection? Ans. Increase 7 mils.

Only five or six misprints were noted.

ALBERT A. BENNETT.

UNIVERSITY OF TEXAS,
May, 1919.

A First Course in Calculus. By W. P. MILNE and G. J. B. WESTCOTT. Part 1: Powers of x. London, G. Bell & Sons Ltd., 1918. 12mo. 20 + 196 pp. Price 3s. 6d.

This little book (namely part 1) duodecimo, and only about half an inch thick, covers roughly a half of what is usually found in American texts on the calculus. The treatment is very well graded, the important ideas are approached gradually and developed at length. Long or troublesome proofs are avoided throughout.

This is made possible by deferring or omitting such topics as the differentiation and integration of transcendental analytic functions, the general evaluation of indeterminate forms, the evolute and involute, general criteria for maxima and minima, mean value theorems, partial differentiation, envelopes, series, singular points and asymptotes, hard definite multiple integrals, and the like; secondly, by explicitly assuming formulas without proof in certain cases, as in physical formulas, and geometric formulas, for example that for radius of curvature, etc.; thirdly, by outlining rather than completing the method of proof in a few instances in which the details of rigor would not appeal to the pupil. The authors state in the preface "there is no attempt whatever to discuss the philosophy of 'limits' and such like" and "It is hoped, therefore, that the treatment will be found suitable for boys and girls in the upper classes of the Secondary Schools, for the pupils in Army Classes, and for students in the Technical schools and colleges."

The book is exceedingly rich in interesting and suggestive exercises, a feature characteristic of most British texts. These exercises are perhaps even too numerous to be useful. They show a strong inclination to be "practical" as is evidenced by the fact that coefficients are seldom selected as integers, and results do not "come out exact" but are usually desired correct to two decimal figures, and that instead of remaining within the subject of algebra, they roam over all branches of elementary physical science. A student completing the course will have not only a facility in handling elementary derivatives, definite integrals and a few types of differential equations, but should have a fair grasp of the significance of many of the laws of mechanics. The print and the make-up of the book are attractive and misprints appear to be rare. The figures are poorly drawn; the perspective ellipses, pages 134, 135, are obviously drawn as circular lunes with sharp corners. The parabola (Fig. 68) has a point of inflection.

A complete set of answers and a usable index are found in the back of the book, and an historical introduction of an entertaining nature opens the text. The wholesale acceptance by the authors of J. M. Child's view that Barrow was the discoverer of the calculus is interesting, but the authors do not pretend to attempt to lend scholarly weight to this view, nor to enter into polemics. They accept what has appeared in print without being sufficiently interested to give the exact reference for their quotation. The book is one of a series of which the best known to American readers is undoubtedly Sommerville's *Elements of Non-Euclidean Geometry*.

It is unfortunate that in their quest for methodological excellence, the authors have paid so little regard to questions of exactness in phraseology. One can hardly read a half dozen consecutive pages without noting some instance of looseness in the language or notation employed. This lack of precision while always irritating is in most of the individual instances trifling, and merits mention only on account of its frequency. Unlike the situation familiar to us in numerous elementary American text-books, the current ambiguity is usually avoided in the more important definitions and explanations. But one or two instances need be cited of features almost characteristic of the book. After giving a satis-

factory definition of maximum, and after having previously treated irregular physical curves, the statement is made (page 30), "then at points on the graph at which either y is a maximum or a minimum, the tangent is parallel to the axis of x." The possibility of a maximum being either a corner point or an end value, while illustrated in figures used in other connections (*e. g.*, Fig. 27, Fig. 46 and Fig. 57) is not even acknowledged throughout the whole book. Again, the sixth chapter bears the title "Differential Equations and Indefinite Integrals," but except for their place in the title the terms "integral," and "indefinite integral" do not appear in the text of the chapter, although assumed as familiar, page 68, example 9.

The term "derivative" appears nowhere in the book, the phrase "differential coefficient" being defined instead, although this in turn is casually referred to as "the differential" in example 40, page 35. The notion of a differential as distinguished from a derivative, according to the usage familiar to American readers, is never suggested.

It has been a tradition that even an approach to the meaning of a differential equation is in the nature of things inaccessible to a student who has not been drilled in so elementary a subject as trigonometry. This tradition is completely ignored in the present little treatise. Here a student with but the rudiments of algebra may easily acquire a grasp of what is meant by a differential equation, a definite integral and other fundamental terms, although applying these only to the "powers of x." It is true that the free use of technical nomenclature in some of the exercises might more than appal the American college freshman whose mathematical foundations have carefully shielded him from the harsh realities of "applications." Just how successfully from the pedagogical viewpoint the subject of the calculus may be split in a not unfamiliar fashion according to the functions used rather than the fundamental concepts employed, can only be examined in the light of the whole subject, and so far as this work is concerned judgment must be withheld since part two is not yet published. The efficiency of the part here at hand however raises squarely the issue as to whether the claims of the calculus as a freshman subject can be ignored even in institutions whose students are generally poorly prepared.

To the attention of all college teachers interested in the closer relationship between elementary mathematical technique and physical applications, as taught with simplicity and directness, this little book is recommended.

ALBERT A. BENNETT.

Des phénomènes gyroscopiques et de leurs principales applications à la navigation. By A. LUCAS. Paris, Challamel, 1918. Royal 8vo. 110 pp. Price in boards 7.80 francs.

Contents—Chapter I, Préliminaires, 1–20: Projections de la vitesse et de l' accélération d'un mobile sur trois axes rectangulaires; Equations du mouvement d'un point matériel et d'un corps solide libre; Mouvement d'un corps solide par rapport à des axes animés d'un mouvement de translation avec lesquels il est entraîné; Rotations; Composition des rotations; Problème sur le mouvement d'un corps ayant un point fixe; Moments d'inertie; Mouvement d'un corps

solide autour d'un axe fixe. Chapter II, Corps de révolution en rotation autour de son axe, 21–38: Problème sur les projections de l'accélération d'un point matériel; Effets produits par les accélérations; Etude de l'effet combiné de deux rotations; Généralisation pour un corps libre de révolution; Effets des forces qui agissent sur un corps en rotation; Etude analytique de l'effet du couple P; Cas ou le corps de révolution a un point de son axe fixe. Chapter III, Applications, 39–107; Gyroscope de Foucault; Gyroscope-marin de E. Dubois; Boussole gyroscopique de Foucault; Compas gyroscopique; Perturbations produites à bord sur l'appareil; Influence de la variation de vitesse du tore; Horizontalité de la rose; Amortissement; Compas gyroscopique Sperry; Emploi du gyroscope sur les torpilles; Projectile oblong; Toupie ordinaire; Balance gyroscopique; Toupie gyroscopique; Gyroscope-collimateur de l'amiral Fleuriais; Effet des frottements; Effet des mouvements du navire; Effet de la rotation de la terre; Utilisation des principes précédents pour déterminer à bord avec un sextant la hauteur d'un astre; Erreur d'horizontalité ou collimation du repère; Autre procédé d'observation; Mouvement de la terre autour de son centre de gravité; Effets gyroscopiques sur les turbines; Gyroscope stabilisateur.

Rehabilitation Monographs. Joint Series No. 26. Unit Course.—Mathematics I. Use of the Slide Rule. [By W. E. BRECKENRIDGE.] Issued by the Federal Board for Vocational Education in coöperation with the Surgeon General's Office and the Bureau of War Risk Insurance. February, 1919. (Trial Edition.) Washington, Government Printing Bureau. 27 pp.

Extract from the preface: "For the purpose of insuring a continuous program of education for wounded and sick soldiers during the time they are in the general hospitals and after their discharge the Surgeon General's Office and the Federal Board for Vocational Education coöperated in the preparation of a series of courses of study. These courses are tentative and suggestive. They may, however, be accepted as models so far as their form and principles which govern their organization are concerned, for all courses whether academic or vocational, where the adjustment of materials is intended to be made to suit the individual capacities and attainments of students."

NOTES.

Il Bollettino di Matematica, 1917–18 (anno 15), contains a sketch of Giuseppe Veronese by P. Gazzaniga (pages 53–65), and an Italian translation with elaborate notes by M. Domenico, of the first of J. W. Young's lectures on *Fundamental Concepts of Algebra and Geometry* (pages 161–183). The complete translation of this work into Italian was referred to in our issue for June.

Vuibert, of Paris, has announced that publication of the following mathematical periodicals is to be resumed this month: *L'Education Mathématique* (which last appeared in July, 1914) and *Revue de Mathématiques Spéciales* (last published in September, 1914).

In our issue for April we had occasion to refer to the publication of the first five parts of *Materialien für eine wissenschaftliche Biographie von Gauss*, edited by Klein, Brendel and Schlesinger. In the latter part of 1918 a sixth part, of 46 pages, by P. Maennchen, was published. It was entitled: *Die Wechselwirkung zwischen Zahlenrechnen und Zahlentheorie bei C. F. Gauss.*

To take the place of *Minerva*, long sent out from Germany, Gauthier-Villars advertises the first volume of an annual, published under the direction of the mathematician Dr. R. de MONTESSUS DE BALLORE, with the following title: *Universitatum et Eminentium Scholarum, Index Generalis. Annuaire Général des Universités. The Yearbook of the Universities.* (Paper, 18 francs; bound, 21 francs.)

The second edition, revised and enlarged, of F. CAJORI's *History of Mathematics* was published in August (Macmillan, price $4.00)—Teacher's Manual, *First Course in Algebra* by W. B. FORD and C. AMMERMAN appeared in July (Macmillan, price $2.00)—Recently announced thoroughly revised editions: *College Algebra* by H. L. RIETZ and A. R. CRATHORNE (Holt, price $1.60) and *Complete School Algebra* by H. E. HAWKES, W. A. LUBY and F. C. TOUTON (Ginn, price $1.40)—In July the University of Chicago Press published E. R. Breslich's *Correlated Mathematics for Junior Colleges* (price $1.25), "comprising college algebra, analytic geometry and some differential calculus." This work is the fourth volume in Breslich's course of mathematics for high schools—In April Heath brought out *New High School Arithmetic: Academic, Industrial, Commercial* by W. WELLS and W. HART (price $1.20).

The Education of Henry Adams. An autobiography, was privately printed, to the number of one hundred copies, in 1907. In September 1918 it was published for the general reader by the Massachusetts Historical Society and became one of the most notable books of the year. The following quotations are made;

"In any and all its forms, the boy detested school, and the prejudice became deeper with years. He always reckoned his school days, from ten to sixteen years old, as time thrown away. Perhaps his needs turned out to be exceptional, but his existence was exceptional. Between 1850 and 1900 nearly everyone's existence was exceptional. For success in the life imposed on him he needed, as afterwards appeared, the facile use of only four tools: Mathematics, French, German, and Spanish. With these he could master in very short time any special branch of inquiry, and feel at home in any society" (page 38).

"The four years passed at college were, for his purposes, wasted. Harvard College was a good school, but at bottom what the boy disliked most was any school at all. He did not want to be one in a hundred—one per cent. of an education. He regarded himself as the only person for whom his education had value, and he wanted the whole of it. He got barely half of an average. In the one branch he most needed—mathematics—barring the few first scholars, failure was so nearly universal that no attempt at grading could have had value, and whether he stood fortieth or ninetieth must have been an accident or the personal favor of the professor. Here his education lamentably failed. At best he could never have been a mathematician; at worst he would never have cared to be one; but he needed to read mathematics, like any other universal language, and he never reached the alphabet" (pages 59–60).

"Socially or intellectually, the college was for him negative and in some ways mischievous. . . . The habit of looking at life as a social relation—an affair of society—did no good. It cultivated a weakness which needed no cultivation. If it had helped to make men of the world or give the manners and instincts of any profession—such as temper, patience, courtesy, or a faculty of profiting by the social defects of opponents—it would have been an education better worth having than mathematics or languages" (page 65).

"He supposed that, except musicians, everyone thought Beethoven a bore, as every one except mathematicians thought mathematics a bore" (page 80).

"Few men in Washington cared to overstep the school conventions, and the most distinguished of them, Simon Newcomb, was too sound a mathematician to treat such a scheme seriously. The greatest of Americans, judged by his rank in science, Willard Gibbs, never came to Washington, and Adams never enjoyed a chance to meet him. . . . By chance it happened that Raphael Pumpelly . . . fell to talking with Adams about these matters [race and sex], and said that Willard Gibbs thought he got most help from a book called the *Grammar of Science* by Karl Pearson. To Adams's vision, Willard Gibbs stood on the same plane with the three or four greatest minds of his century, and the idea that a man so incomparably superior should find help anywhere filled him with wonder. He sent for the volume and read it. . . . Here came in, more than ever, the fatal handicap of ignorance in mathematics. Not so much the actual was needed, as the right to judge the product of the tool" (pages 377, 449).

ARTICLES IN CURRENT PERIODICALS.

AMERICAN OXONIAN, Concord, N. H., volume 6, no. 2, April, 1919: "The Choice of a course at Oxford: Mathematics" by R. W. Burgess, 60–62; [This article is reprinted from the issue for January, 1916. Quotation: "The young American who desires to study mathematics and prepare himself for the work of a college teacher in that subject may find in Oxford and the Rhodes Scholarship a valuable opportunity to secure a part of his preparation. In considering this possibility he should not be misled by the fact that the subjects stated in the *Handbook* are rather limited in number and in part are those already begun in his previous work. He must realize that the study of mathematics at Oxford is persisted in only by the select few, almost without exception men on scholarships obtained after a competitive examination which makes it almost certain that these men know more mathematics than an American whose college work has included the first course in calculus, and are far superior to him in problem-solving."]

ATHENAEUM, London, May 9, 1919: "The mind of Leonardo," 307–308. [Leonardo da Vinci died on May 2, 1519. He anticipated by about "four centuries and a half the invention of tanks . . . Mathematics, pure and applied, anatomy, geology, botany, astronomy—Leonardo traversed almost the whole domain of the sciences. Wherever he looked, his clear unprejudiced eyes discovered something to which his contemporaries were blinded by ancient habits of thought."]—May 16: "On relative motion," 337.

BOLLETTINO DI BIBLIOGRAFIA E STORIA DELLE SCIENZE MATEMATICHE, year 20, 1918, no. 2, April–June: "Adolfo Viterbi (1873–1917)" by G. Vivanti, 33–37. [With portrait and list of 33 of Viterbi's publications in applied mathematics.] "Eugenio Elia Levi" (1883–1917) by G. Fubini and G. Loria, 38–45. [With portrait and list of 34 of Levi's publications in pure mathematics.] Review by E. Bortolotti of Pincherle's *Lezioni di calcolo infinitesimale* (Bologna, 1915), 46–51; Review by E. Ciani of Enriques's *Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche*, volume 1 (Bologna, 1915), 52–62.

BULLETIN DES SCIENCES MATHÉMATIQUES, 2d series, volume 43, 1919, January: Review by G. J[ulia] of Borel's *Leçons sur les fonctions monogènes uniformes d'une variable complexe* (Paris, 1917), 5–9; Review by L. Godeaux of Enriques's *Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche*, volume 2 (Bologna, 1918), 10–12; "Démonstration directe du dernier théorème de Poincaré" by E. Gau, 12–17; "Sur un point d'histoire des groupes finis discontinus" by G. A. Miller, 17; "Sur certains systèmes d'équations de Fredholm" by H. Villat, 18–24—February: "Sur certains systèmes d'équations de Fredholm" (concluded) by H. Villat, 25–48.

EDUCATIONAL ADMINISTRATION AND SUPERVISION, Hershey, Pa., volume 5, no. 3, March, 1919: "Mathematics in the education of youth from 12 to 18 years—some proposals for further study" by D. Snedden, 125–144.

ELEMENTARY SCHOOL JOURNAL, Chicago, volume 19, no. 8, April, 1919: "A series of diagnostic tests in arithmetic" by W. S. Monroe, 585–607.

L'ENSEIGNEMENT MATHÉMATIQUE, volume 20, no. 4, June, 1919: "Les noms et les choses, remarques sur la nomenclature mathématique" by G. Loria, 237–243: "Les origines d'un problème inédit de E. Toricelli" by E. Turrière, 245–268—[The problem: If a, b, c are the sides of a triangle and $a^2 = b^2 + c^2$, $a = \square$, $b + c = \square$, $a + b = \square$ $(b > c)$, find a, b, c. Of the infinite number of solutions of a closely related earlier problem by Fermat the smallest numbers were those he gave: $a = 4\,687\,298\,610\,289$, $b = 4\,565\,486\,027\,761$, $c = 1\,061\,652\,293\,520$ $a + c = (2\,397\,697)^2$. Euler discussed the problem also;] "Remarque sur l'integrale $\int w\,dx$" by M. Petrovitch, 268–270; "Combinaisons déterminantes" by J. Helmis, 271–276; "Sur la détermination et quelques propriétés des lignes élastiques" by M. Zack, 276–285; "Sur les équations transcendantes qui se présentent dans la théorie des tiges élastiques" by M. Paschoud, 286–292; "Mélanges et correspondance" by M. d'Ocagne and H. Brocard, 292–293; "La collaboration scientifique internationale" 294–298; "Société suisse des professeurs de mathématiques," 298–302; "Nécrologie," 305–308; Review by A. Buhl of Appell and Dautheville's *Précis de mécanique rationnelle* (2e éd. Paris, 1918), 309–310; Review by R. Masson of Du Pasquier's *Introduction à la science actuarielle* (Neuchâtel, 1919), 311–302; Review by H. Fehr of Fricke's *Lehrbuch der Differential- und Integralrechnung und ihrer Anwendungen* (2 vols., Leipzig, 1919), 312.

HERMATHENA, Dublin, volume 18, no. 41, April, 1919: "Geometry of complex variables" by R. Russell, 204–239.

MESSENGER OF MATHEMATICS, volume 48, no. 3, July, 1918: "On a plane configuration of points and lines connected with the group of 168 plane collineations" by W. Burnside, 33–34;

"The conjoint method of factorization of $N = xy$, by ascent and descent upon x, the process being, in each part, alike in principle" by D. Biddle, 35–46; "On the probable regularity of a random distribution of points" by W. Burnside, 47–48.

NATURE, volume 103, April 17, 1919: Review by G. Greenhill of Gray's *A Treatise on Gyrostatic and Rotational Motion* (London, 1918), 121–122.

NOUVELLES ANNALES DE MATHÉMATIQUES, volume 78, no. 1, January, 1919: "Note sur le déplacement infiniment petit d'un trièdre attaché a une courbe" by M. Leveugle, 1–12; "Problème sur les coniques posé par la metrophotographie" by M. d'Ocagne, 12–14; "Sur une transformation géométrique" by M. F. Egan, 14–18; "Applications d'une formule de géométrie infinitésimale" by F. Balitrand, 19–21; "Condition de convergence de trois droites de Simson" by P. Boulanger, 22–24; [Finds the necessary and sufficient condition that the Wallace lines of three points of a circle circumscribing a triangle are concurrent.] "Solution d'un exercice proposé aux candidats a l'agrégation par les *Nouvelles Annales de Mathématiques* (1914, p. 522)" by L. Long, 25–31; Questions and solutions, 31–40—No. 2, February: "Sur les droites coupant une conique sous un angle donné" by R. Bouvaist, 41–49; "Sur les spirales sinusoïdes osculantes" by R. Goormaghtigh, 49–54; "Sur des séries remarquables" by M. Weill, 54–58; "Sur un problème de probabilité" by J. Haag, 58–69; Questions and solutions, 69–80.

OPEN COURT, volume 33, no. 4, April, 1919: "The Mittag-Leffler Testament and Institute," 249; Frontispiece portrait of Mittag-Leffler, drawing by C. W. Maud "after a photograph in the *American Journal of Mathematics.*" [No such photograph appeared in the *Journal.*]

RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO, volume 42, May–December, 1917 (January, 1919): "Integral invariants in projective geometry" by E. J. Wilczynski, 128–137; "Certain surfaces of Voss and surfaces associated with them" by L. P. Eisenhart, 145–166; "The characteristic numbers of a real algebraic plane curve" by J. L. Coolidge, 260–266.

REVUE GÉNÉRALE DES SCIENCES PURES ET APPLIQUÉES, volume 30, March 30, 1919: Review of Lecornu's *La mécanique. Les idées et les faits* (Paris, 1918), 184–185—April 30: Review by M. d'Ocagne of *Oeuvres de G. Halphen*, tome 2 (Paris, 1918), 251.

REVUE SCIENTIFIQUE, volume 57, April, 19–26, 1919: "Sur le principe de relativité" by R. Chevassus, 230–233—May 3–10: Review by R. Garnier of Goursat's *Cours d'analyse mathématique*, tome 2 (Paris, 1918), 284.

SCHOOL SCIENCE AND MATHEMATICS, volume 19, June, 1919: "Proving a geometrical fallacy by trigonometry" by W. W. Johnson, 527–528 [In connection with the title of the article a reference is given to John Phin, *The Seven Follies of Science*, pp. 126–7; see also W. W. R. Ball, *Mathematical Recreations and Essays*, 1911, pp. 52–53, and this MONTHLY, May, 1918, p. 236], "Concerning psychological tests in the army and their meaning for the teacher" by J. W. A. Young, 544–548 [Quotation: "No teacher of mathematics who believes that the study of his subject can contribute to the advancement of the pupil's general intelligence and who teaches with such advancement as one of his aims, need be disquieted by any assertions that psychology is against him. Even though he may see no whit of improvement in the pupil's mathematical thinking, or in his ability to handle the subject's more complicated processes, or in his appreciation of its wide range of usefulness, if the teacher succeeds only in giving the pupil such increased familiarity with the most elementary notions of the subject as will enable him with greater speed and accuracy to mark a cross in the square and not in the triangle, and to draw a line from the second circle above the third and below the fourth to the fifth, he may rest content with the knowledge that the best psychologists that the United States Government could summon to its aid will, in consequence of that improvement, rate that pupil higher in *general intelligence.*"] Problems and solutions, 563–567 [these include a discussion of Morley's theorem].

SCIENCE, new series, volume 49, May 23, 1919: "The history of science" by George Sarton, 497 [Letter with reference to Dr. Felix Neumann's article in Science for April 4.]—June 6: "Gabriel Marcus Green" by W. F. Osgood, L. Wiener, and D. Jackson, 534–535 [Minute on the life and services of Dr. Green placed upon the records of the faculty of arts and sciences of Harvard University, at the meeting of April 1, 1919.]

TIDSKRIFT FÖR ELEMENTÄR MATHEMATIK, FYSIK, OCH KEMI, Stockholm, volume 2, no. 1, September, 1918: "En reciprocitet hos trianglar" by N. Lindskog, 1–13; "Om lösningen av fjärdegradsekvationer" by O. Josephson, 14–17; "En geometrisk sats, löst med komplexa tal" by D. Larsson, 18–19; "Ett elementärt bevis för minimum av avvikning vid ett prisma" by A. Meyer, 20–21; Mathematical questions and solutions, 58–62—No. 2, December: Något om förvandlingen av allmänna bråk till decimalbråk" by A. Lindhagen, 65–67; "Om den ptolemeiska

satsen" by V. Lenander, 68–71; "Relativitetsprinciperna" by H. Fixén, 82–100; Mathematical questions and solutions, 118–128—No. 3, March, 1919: "Om seriesummeringar" by A. Meyer, 129–158; "Tillämpad matematik i skolans astronomikurs" by C. Lönnquist, 159–173; Mathematical questions and solutions, 181–192.

TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY, volume 20, no. 2, April, 1919: "Memoir on the general theory of surfaces and rectilinear congruences" by G. M. Green, 79–153; "Modular concomitant scales, with a fundamental system of formal covariants, modulo 3, of the binary quadratic" by O. E. Glenn, 154–168; "Concerning a set of postulates for plane analysis situs" by R. L. Moore, 169–178; "On the limit functions of sequences of continuous functions converging relativity uniformly" by E. W. Chittenden, 179–184.

AMERICAN DOCTORAL DISSERTATIONS.

E. D. GRANT, 1873– , *Motion of a flexible cable in a vertical plane.* Lancaster New Era, 1918. 4to. 28 pp. (Chicago, 1916).

J. E. MCATEE, —— – 1918, *Modular invariants of a quadratic form for a prime power modulus.* [Reprinted from *American Journal of Mathematics*, volume 41, 1919]. Pp. 225–242. (Chicago, 1917).

UNDERGRADUATE MATHEMATICS CLUBS.

EDITED BY U. G. MITCHELL, University of Kansas, Lawrence.

CLUB ACTIVITIES.

DENISON MATHEMATICS CLUB, Denison University, Granville, Ohio.
[*1918*, 403–04.]

The Denison Mathematics Club had a somewhat unique scheme for their first annual banquet which was held on May 20, 1919. The Club walked in a group about a mile to a farmhouse where the meal was served. The twenty-five who attended were seated at two long tables. In the center of each table was a vase holding a large bouquet of various mathematical symbols, cut from white paper, each having a stem of slender wire. Cardioids cut from green paper and mounted in the same way supplied the leaves. The candle shades were encircled with simple equations cut in the cardboard of the inner shade which formed the support for the outer tissue shade, so that, when the candles were lighted, these equations stood out distinctly from their darker surroundings. Other symbols were suspended by means of fine thread from the lower edges of the shades.

Below is given a cut of the menu and program pages.

Since the reader can not have the opportunity of checking his interpretation of the menu by the food served the following key is given:

Course 1.

n_1: Escalloped potatoes.
n_2: Sliced veal loaf.
n_3: Rolls with butter.
n_4: Olives.
n_5: Coffee with cream and sugar.

(The two things selected in advance were the water and the the coffee berry.)

Course 2.

n_1: Fruit salad with dressing poured over the top and the whole "over" a lettuce leaf.

n_2: Wafers.

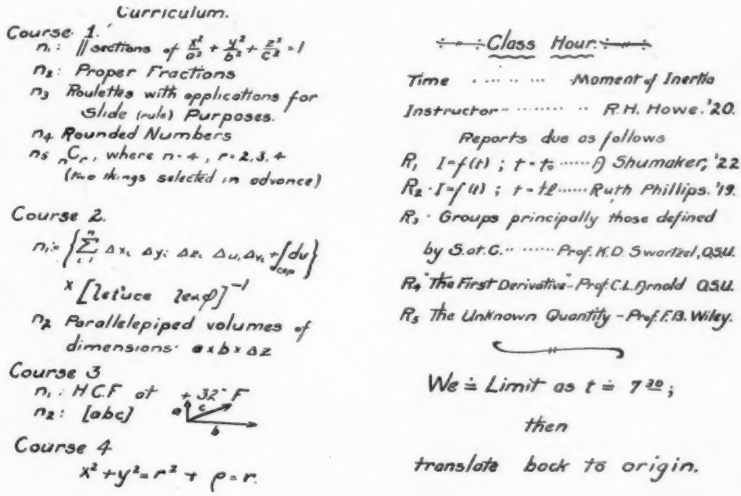
Course 3.

n_1: Ice cream.

n_2: Cakes as parallelopipeds.

Course 4.

Mints.


Curriculum.
Course 1.
Course 2.
Course 3
HCF of
+32° F
Course 4
Class Hour.
Time ..... Moment of Inertia
Instructor ..... R. H. Howe. '20.
Reports due as follows
R1 I=f(t) ; t=t0 ..... J. Shumaker, '22
R2 I=f(u) ; t=t2 ..... Ruth Phillips. '19.
R3 Groups principally those defined
by S. of C. ..... Prof. K. D. Swartzel, OSU.
R4 The First Derivative - Prof. C. L. Arnold OSU.
R5 The Unknown Quantity - Prof. F. B. Wiley.
We = Limit as t = 7 29 ;
then
translate back to origin.

The guests of honor were Professors Karl D. Swartzel and Charles L. Arnold of Ohio State University. Professor Swartzel was a district examiner for the S. A. T. C. which fact suggested the topic for his toast. Professor Arnold is the main sponsor of the mathematics club at the Ohio State University.

The copy of the program given above was lettered by Mr. Richard H. Howe '20, president of the Club, who acted as toastmaster at the banquet.

THE MATHEMATICS CLUB OF NORTHWESTERN UNIVERSITY, Evanston, Ill.
[*1918*, 132–134, 409.]

The officers of the Club for the year 1918–19 were: President, Franklin Mohr '18; vice-president, Frank Danielson '18; secretary, Helen Kelly '19; treasurer, Margaret Walker '19; faculty adviser, Mrs. Mayme Irvin Logsdon, instructor.

The following programs were given during the year.

September: "Some mathematics of artillery fire" by Theodore Doll Gr.; Informal talk on radio telegraphy, by Franklin Mohr '18.

October: "The game of nim" by Mrs. Mayme Irvin Logsdon, Instructor.

November: "Mercator's projection" by William H. Burger, Professor of Civil Engineering.

December: "Applications of algebra to arithmetic" by Margaret Walker '19.

January, 1919: "Finger-counting among the ancients" by Christine McMartin '19; "A short-hand system for advanced mathematical analysis" by Gladys Williams '19.

February: "Rotating fluid bodies" by Professor Elton J. Moulton.

March: "Integral right triangles" by Helen Kelly '19; "Fermat's last theorem" by Professor David R. Curtiss.

April: "Some indefinite locations at sea" by Leonard Janes '17.

May: "Communication with other planets by means of mathematics" by Eleanor Olmstead '19; "Problem of two bodies under the action of gravity" by William Janes '19.

Near the close of the year, a very pleasant social gathering was held at the home of Professor Curtiss. The reading of humorous sketches and participation in guessing contests furnished a good part of the entertainment.

THE PENTAGRAM, University of Texas, Austin, Texas. [*1918*, 273–274.]

The *University of Texas Bulletin* for May 10, 1919, gives the following resume of the Pentagram's activities during the year 1918–19:

"The Pentagram, the mathematics club at the University of Texas, is in its third year. It has proved itself to be profitable and stimulating to all its members. The influenza epidemic caused the suspension of the activities of the club the greater part of the first half of the present academic year. The work was resumed, however, about the middle of the winter term and the usual interest is being taken by both students and faculty. Under the leadership of Professor Albert A. Bennett, who organized the club in 1916, and who has returned from government service, the Pentagram looks forward to even greater success than it has had thus far.

"The officers of the Pentagram for the year are: President, Bert McDonald '19; vice-president, Georgie Savage '20; secretary-treasurer, Goldie P. Horton, instructor in pure mathematics; student member of the executive council, Warren Simonds '20; faculty member of the executive council, Professor Albert A. Bennett."

The meetings of the year were as follows:

November 15, 1918: "Derivativeless continuous functions" by Professor M. B. Porter.

January 9, 1919: "Homogeneous coördinates" by Professor J. W. Calhoun.

February 18: Social meeting.

March 5: "The graph of $y = f(x)$ for complex values" by Alice Ballard '21.

March 21: "Continued fractions" by Ernest Normand '18.

April 4: "A comparison of formulas used in computing parabolic arches" by Warren Simonds '20.

THE MATHEMATICS CLUB OF GOUCHER COLLEGE, Baltimore, Maryland.
[1918, 357–358.]

During the year 1918–19, Professor Florence P. Lewis, who has heretofore had charge of the Mathematics Club at Goucher, was Exchange Professor at Wellesley College and Misses Margaret Amig '19 and Marguerite Lehr '19 acted as program committee. Below is given a complete list of the programs given during the year.

December 16, 1918: "Report of the summer meetings of the Mathematical Association of America and the American Mathematical Society" by Professor Clara E. Smith, Exchange Professor from Wellesley College.

January 6, 1919: "Report of a meeting of the New England Association of Teachers of Mathematics" by Professor Florence P. Lewis.

January 20: "Map-making" by Marie Whelan '18.

February 3: "Empirical equations" by Professor Clara L. Bacon.

February 19: "Paper folding" by Margaret Amig '19.

March 5: "Persons and anecdotes in mathematics" by Ethel Carrol '20.

March 17: "History of the squaring of the circle" by Loretta Whelan '20; "Simple Hints on plotting graphs" by Varina Davis '20; "Magic squares" by Deldee Groff '20.

April 14: "Russian peasant method of counting" by Eleanor Norris '20; "The planimeter" by Louise Ellery '20 and Mary Brower '20.

May 5: "The golden mean" by Jean Burke '20; "Zeno's paradoxes" by Ethel Fox '19.

May 13: Picnic.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

Send all communications about Problems to B. F. FINKEL, Springfield, Mo.

NOTE ON THE NUMBER OF SOLUTIONS OF LINEAR INDETERMINATE EQUATIONS.

BY D. N. LEHMER, University of California.

The solution of problem 2700 on page 215 of the May issue of the MONTHLY suggests the question as to whether a process may be found for obtaining, at least approximately, the number of solutions of an indeterminate equation in several variables. In the *Proceedings of the National Academy of Sciences*, April, 1919, I have given a method for getting, by a simple and direct process, the general form of the solution of any linear indeterminate equation in any number of variables. Such equations have been solved by Jacobi (*Werke*, Vol. 6, p. 355) but his process does not lead to the general form directly.

One finds for the equation $6w + 4x + y = 875$, which is the one treated in the Artemas Martin problem, the following general solution:

$$x = X - Y,$$

$$y = 875 - 4X - 2Y,$$

$$w = Y;$$

which give, since $w + x + y + z = 250$,

$$z = 3X + 2Y - 625.$$

All values of x, y, z, w, may be obtained from these by assigning integral values to X and Y. In order that x, y, z, w, shall be positive certain restrictions must be put on the values of X and Y. These restrictions are most easily understood when the pairs (X, Y) are taken as the coördinates of a point in the plane. It thus appears that the points (X, Y) available for our problem must lie in or on the boundaries of a quadrilateral enclosed by the four lines: $Y = 0$, $X - Y = 0$, $875 - 4X - 2Y = 0$, $3X + 2Y - 625 = 0$. The number of points with integer coördinates in this region will be approximately equal to the area of the quadrilateral which is found to be 2930. If Mr. Uhler's enumeration is correct this result is in error by 36. The actual number might be obtained by counting the number of lattice points in this quadrilateral.

The method applies as easily to problems involving a greater number of variables. Instead of areas one will have to consider volumes in space of three or more dimensions.

PROBLEMS FOR SOLUTION.

2784. Proposed by T. H. GRONWALL, New York City.

Show that all solutions in integers of $y^2 = 1 + x + x^2 + x^3 + x^4$ are given by

$$\begin{array}{lccc} x = & -1, & 0, & 3; \\ y = & \pm 1, & \pm 1, & \pm 11. \end{array}$$

2785. Proposed by W. H. ECHOLS, University of Virginia.

If on the sides, as bases, of any closed plane polygon, there be constructed similar triangles similarly placed, all outward or all inward, then the centroid of the vertices of these triangles coincides with the centroid of the corners of the polygon.

2786. Proposed by R. A. JOHNSON, Hamline University.

At what angle should the face of a golf club (driver) be laid back, in order to secure the maximum distance of flight? Show that, under certain assumptions, the angle should be 15°.

2787. Proposed by WARREN WEAVER, University of Wisconsin.

In the gambling game known as "craps," two dice are thrown. The one throwing the dice makes a bet which is covered by an equal amount by one or more players opposing him. If he throws on the first throw a sum of seven or eleven he wins at once; if he throws a sum of two, three, or twelve he loses at once; if he throws any other sum he continues throwing until he either duplicates his original sum, and wins, or throws a sum of seven and loses. Show that the probability that the one who is to throw the dice will win is 0.49847, so that the game would be very nearly fair if one person were to throw the dice continuously instead of changing about, as is actually done.

Note.—Compare the article on "Probabilities in the game of 'Shooting Craps'" published elsewhere in this issue of the MONTHLY.—Editor.

SOLUTIONS OF PROBLEMS.

245 (Number Theory) [May, 1916; May, 1917]. Proposed by NORMAN ANNING, Chilliwack, B. C.

Show that $x^2 + y^2 = (a_1 a_2 \cdots a_m)^n$ has $4(n+1)^m$ solutions in integers, in 2^{m+2} of which x and y are relatively prime, the a's being primes of the form $4k+1$ and n a positive integer.

I. Solution by Frank Irwin, University of California.

In Dirichlet's *Zahlentheorie* (4th edition, page 164) it is shown that the equation $x^2 + y^2 = M$ has 2^{m+2} solutions in integers x, y relatively prime, where M has m different prime factors (all of them being of the form $4k+1$). It may be expressly noted that the solutions, if any, in which x or y are zero are not counted in the number 2^{m+2}.

The solutions of our given equation in which x, y are not relatively prime may evidently be obtained by taking relatively prime solutions of equations of the form $x'^2 + y'^2 = N$, where N is the quotient of the right side of our equation by any square integer d^2; then $x = dx'$, $y = dy'$. The number of these solutions may be found as follows:

(i) n odd. N may contain a_i 1, 3, $\cdots$ or n times, $i = 1, 2, \cdots m$, and for each such choice of N we have 2^{m+2} solutions; altogether, then, $[(n+1)/2]^m 2^{m+2} = 4(n+1)^m$ solutions.

(ii) n even. Here N need not contain all the m a's as factors, and we must consider separately the cases thus arising. If N has m different prime factors, we get $(n/2)^m 2^{m+2}$ solutions; if N has $m-1$ different prime factors, we get $m(n/2)^{m-1}2^{m+1}$ solutions; if N has $m-2$ different prime factors, we get $\binom{m}{2}(n/2)^{m-2}2^m$ solutions; etc., etc.; that is, in all

$$4\left[n^m + mn^{m-1} + \binom{m}{2} n^{m-2} + \cdots + 1\right] = 4(n+1)^m$$

solutions.

II. Solution by C. F. Gummer, Queen's University.

It is well known (see, for instance, Barlow's *Theory of Numbers*, Chapter IX) that for every prime a_i of the form $4k+1$ positive integers x_i, y_i, relatively prime, may be found such that $x_i^2 + y_i^2 = a_i$. Let the even one be x_i, and let us write $x_i = \sqrt{a_i}\cos\theta_i$, $y_i = \sqrt{a_i}\sin\theta_i$. Then we can prove that a solution in integers of

$$x^2 + y^2 = a_1^{n_1} a_2^{n_2} \cdots a_m^{n_m} = A, \tag{1}$$

where the a's are primes of the form $4k+1$, is given by

$$x = \sqrt{A}\cos\Theta, \qquad y = \sqrt{A}\sin\Theta, \qquad \Theta = k\pi/2 + \sum_i r_i\theta_i, \tag{2}$$

where r_i is any one of the integers n_i, $n_i - 2$, $n_i - 4$, $\cdots$, $-n_i$, and k is an integer.

There are no solutions of (1) other than those included in (2). For let $x^2+y^2=x'^2+y'^2=B$, where B is odd, x, y, x', y' positive or zero, x and x' even, and $x > x'$. Since

$$\tfrac{1}{2}(x+x')\cdot\tfrac{1}{2}(x-x') = \tfrac{1}{2}(y'+y)\cdot\tfrac{1}{2}(y'-y),$$

and neither member is zero, we infer that every divisor of $\frac{1}{2}(x+x')$ is a divisor of either $\frac{1}{2}(y'+y)$ or $\frac{1}{2}(y'-y)$, and so for $\frac{1}{2}(x-x')$. Hence, we may write

$$\tfrac{1}{2}(x+x') = pr, \qquad \tfrac{1}{2}(x-x') = qs, \qquad \tfrac{1}{2}(y'+y) = ps, \qquad \tfrac{1}{2}(y'-y) = qr,$$

and consequently

$$x = pr + qs, \qquad y = ps - qr, \qquad x' = pr - qs, \qquad y' = ps + qr,$$

and

$$B = x^2 + y^2 = (p^2+q^2)(r^2+s^2).$$

Since no one of p, q, r, s can be zero, it follows that $x^2 + y^2 = B$ can admit two essentially distinct solutions only if B is composite, and therefore that the θ_i are unique. Also, on writing

$$p = \sqrt{C}\cos\theta, \qquad q = \sqrt{C}\sin\theta, \qquad r = \sqrt{D}\cos\phi, \qquad s = \sqrt{D}\sin\phi,$$

we see that if $x^2 + y^2 = B$ has two such solutions they are deducible from integral solutions of

$x^2 + y^2 = C$ and $x^2 + y^2 = D$ $(CD = B)$ by the rule $x = \sqrt{B}\cos(\theta \mp \phi)$, $y = \sqrt{B}\sin(\theta \mp \phi)$. If we allow for the interchange of x and y and for negative values, the rule becomes

$$x = \sqrt{B}\cos(l\pi/2 \pm \theta \pm \phi), \qquad y = \sqrt{B}\sin(l\pi/2 \pm \theta \pm \phi);$$

and on applying this method successively to A and its divisors we get the formula (2).

We wish to know how many distinct solutions are given by (2). Two solutions given by

$$\Theta' = k'\pi/2 + \sum_i r_i'\theta_i \qquad \text{and} \qquad \Theta'' = k''\pi/2 + \sum_i r_i''\theta_i$$

are equal only if

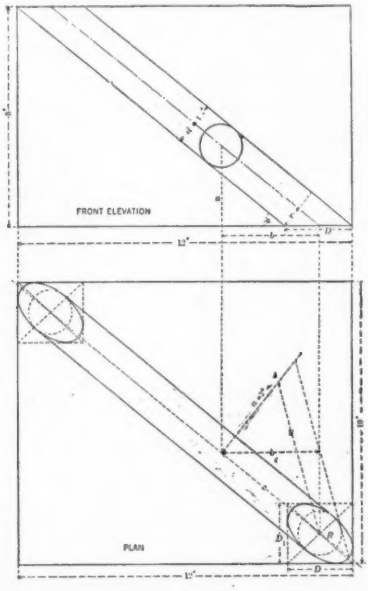
$$(k' - k'')\pi/2 + \sum_i (r_i' - r_i'')\theta_i$$

is a multiple of 2π. If this is so and $r_1' - r_1'' \neq 0$, it follows that

$$\cos(r_1' - r_1'')\theta_1 = \cos\left\{K\pi/2 + \sum_{i=2}^{m} (r_i' - r_i'')\theta_i\right\}.$$

But the left member, being a polynomial of degree $|r_1' - r_1''|$ in $\cos\theta$ with integral coefficients of which the highest is a power of 2, is a fraction with denominator $a_1^{|r_1' - r_1''|}$, while the right member cannot be of this form. Hence $r_1' = r_1''$. Two solutions are then identical only if $r_i' = r_i''$ $(i = 1, \cdots, m)$ and $k' - k''$ is a multiple of 4. The number of effectively distinct combinations of values of $k, r_1, \cdots, r_n$ is therefore $4(n_1 + 1)(n_2 + 1)\cdots(n_m + 1)$.

In some of the solutions given in (2), x and y will admit the common divisor $E = a_1^{\varepsilon_1}a_2^{\varepsilon_2}\cdots a_m^{\varepsilon_m}$. These are found by multiplying by E the roots of the equation $x^2 + y^2 = a_1^{n_1 - 2\varepsilon_1}a_2^{n_2 - 2\varepsilon_2}\cdots$, and therefore occur in the cases where $|r_i| \leqq n_i - 2\varepsilon_i$ $(i = 1, 2, \cdots m)$. The solutions have no common divisor when there is no number of the form E other than unity for which this condition holds, that is when $r_i = \pm n_i$ $(i = 1, \cdots m)$. The number of relatively prime solutions is therefore $4.2^m = 2^{m+2}$.



2696 [April, 1918]. Proposed by L. E. LUNN, Heron Lake, Minnesota.

An air pipe 18 inches in diameter passes diagonally through a room from one lower corner to the opposite upper corner leaving through elliptical openings in the floor and ceiling, so that the ellipses are tangent to two boundaries of the floor and to the two opposite boundaries of the ceiling. If the room is $10 \times 12 \times 8$ feet, find the remaining cubic capacity of the room.

SOLUTION BY A. R. NAUER, St. Louis, Missouri.

Make D the apparent width of the floor or ceiling contact as seen at the front elevation, and D_1 the same for the side elevation. Then

$$D = d/\sin\alpha = -\frac{27}{61.75} + \frac{\sqrt{468 \times 61.75 + 27^2}}{61.75}$$

$$= 2.35025 +$$

and

$$D_1 = -\frac{22.5}{61.75} + \frac{\sqrt{369 \times 61.75 + 22.5^2}}{61.75} = 2.10715 +$$

a is the length of a perpendicular to the floor from an arbitrary point A on the center line of the pipe; a is here taken 3 feet.

r is the radius and d the diameter of the pipe.

R is the half major axis of the bounding ellipses, and

b is the distance, in front elevation from the foot of perpendicular a, to center of D, which is also the center of contact ellipses;

$$b = \frac{a}{8}(12 - D) = \tfrac{1}{8}(9.64975) = 3.61865, \qquad b^2 = 13.095 +$$

c is the distance from the foot of perpendicular a to center of ellipse.
y is the distance from A to center of ellipse.

$$c^2 = b^2 \frac{(12 - D)^2 + (10 - D_1)^2}{(12 - D)^2} = 21.8552 +$$

$$y = \sqrt{a^2 + C^2} = \sqrt{30.8552} = 5.5547 +$$

$$R = \frac{r}{a} y = \tfrac{1}{4} y = 1.3887 +$$

Volume of pipe is $8\pi Rr = 2.51328 \times 1.3887 \times 0.75 = 26.17644 +$ cubic feet.
Hence remaining capacity of room is 960 cu. ft. $-$ 26.17644 cu. ft. equal to 933.82356 cubic feet.

2722 [September, 1918]. Proposed by FRANK IRWIN, University of California.

The number of terms in the general polynomial of the nth degree in m variables and in that of the mth degree in n variables is the same. It would be interesting to devise schemes which, without assuming this result, should exhibit the terms of these polynomials in one-to-one correspondence with each other.

Solution by C. F. Gummer, Queen's University.

Consider first the polynomial $P_n(x_1, x_2, \cdots, x_m)$ of degree n, with coefficients all equal to 1. The general term is

$$x_1^{p_1} x_2^{p_2} \cdots x_m^{p_m} x_{m+1}^{p_{m+1}}, \qquad (\Sigma p = n),$$

where $x_{m+1} = 1$. Let the term be written at length, and a y $(= 1)$ inserted after each group of like x's except after the one for x_{m+1}, the y appearing even when the corresponding p is zero. The term is completely defined by the positions of the y's, so that the subscripts may be dropped, and the term written $xx \cdots yxx \cdots yxx \cdots$. Thus, in a polynomial of degree 4 in 5 variables, $x_1^2 x_3$ will be denoted by $xxyxyyyx$, the last x representing $x_6 = 1$. The various terms of P_n then correspond to the permutations of n x's and m y's. In the same way the terms of P_m $(x_1, x_2, \cdots x_n)$ of degree m may be made to correspond to the permutations of m x's and n y's. We may now put into one-to-one correspondence the terms of P_n and P_m which differ by interchange of the letters x and y.

Since the choice of a term in P_n corresponds to the choice of positions for the m y's, this method furnishes a direct explanation of the fact that the number of combinations of $m + 1$ kinds of thing taking n things at a time and allowing repetition is $\binom{m+n}{n}$.

2729 [November, 1918]. Proposed by N. P. PANDYA, Sojitra, India.

Solve in integers $x^3 + 3y^4 = z^2$.

Solution by S. A. Corey, Des Moines, Iowa.

Having obtained by any means one solution x, y, z, it is easily seen that a^4x, a^3y, a^6z is a solution, where a may be any integer. Since 1, 2, 7 and 1, 1, 2 are solutions, a^4, $2a^3$, $7a^6$ and a^4, a^3, $2a^6$ are solutions, whatever the value of a.

2731 [November, 1918]. Proposed by J. K. WHITTEMORE, Yale University.

A bowl is in the form of a paraboloid of revolution. If for a given volume the surface is a minimum, prove that the ratio of the diameter of the top to the depth is approximately 1.86.

Solution by Elijah Swift, University of Vermont.

Let the parabola have the equation, $y = kx^2$. Call the depth of the bowl, l. The required ratio is

$\frac{2\sqrt{\frac{l}{k}}}{l} = \frac{2}{\sqrt{kl}}$; the volume, $V, = \frac{\pi}{2k} l^2$; the surface, $S, = \frac{\pi}{6k^2}\{(1+4kl)^{3/2} - 1\}$.

The necessary condition for a minimum may be found by equating to zero the partial derivatives of $S + \lambda V$ (λ a parameter) with respect to k and l. We thus obtain, after simplification, $\sqrt{1+4kl}(-2-2kl) + 2 - 3k^2\lambda = 0$, and $\sqrt{1+4kl} + l\lambda = 0$.

Eliminating λ, there results $\sqrt{1+4kl}(-2+kl) = -2$.

Rationalizing and reducing, we obtain a cubic equation with the roots $kl = 0$, $1.1569+$. $2.5931+$. Of these, the last is not a root of the last equation given above, but is extraneous, The first corresponds geometrically to an infinite ratio, *i. e.*, a "flat bowl," or a maximum. If there be a minimum, as seems evident from geometric considerations, it must correspond to the remaining root, $kl = 1.1569+$. The ratio corresponding is $\frac{2}{\sqrt{kl}} = 1.8595$ or $1.86-$.

Also solved by A. M. HARDING, GERTRUDE I. MCCAIN, and the Proposer.

NOTES AND NEWS.

EDITED BY E. J. MOULTON, Northwestern University, Evanston, Ill.

Dr. W. D. MACMILLAN has been promoted from assistant professor to associate professor of astronomy at the University of Chicago.

Mr. PAUL HEMKE, of Northwestern University, has accepted a position as instructor in mathematics at the U. S. Naval Academy at Annapolis.

Dr. OTTO DUNKEL has been promoted from assistant professor to associate professor of mathematics at Washington University, St. Louis.

Dr. L. C. KARPINSKI has been promoted from associate professor to professor of mathematics at the University of Michigan.

Recent appointments as instructors in mathematics at Yale in the Sheffield Scientific School for next year are: Dr. T. M. STETSON, Mr. EUGENE TAYLOR, Mr. HERMAN BETZ.

Captain J. V. MCKELVEY, of Cornell University, has been appointed assistant professor of mathematics at Iowa State College, Ames, Iowa.

Mr. F. L. KERR, of Northwestern University, has resigned his position as instructor in mathematics to become registrar of the university.

At Cornell University, Mr. H. S. VANDIVER and Dr. G. M. ROBISON have been appointed instructors in mathematics.

Owing to an unexpected increase in the number of students, the regular staff of the department of mathematics in the summer session of Cornell University was augmented by the appointment of Professor M. G. GABA of the University of Nebraska, Professor F. W. BEAL of the University of Tennessee, and Mr. ARTHUR HARMAN.

D. H. R. HASSÉ, late fellow of St. John's College, Cambridge, and senior lecturer in mathematics at the University of Manchester, has been appointed professor of mathematics at the University of Bristol.

J. HAAG, who received his doctorate from the University of Paris in 1910, has been appointed professor of differential and integral calculus at the University of Hancy.

It is planned to organize at the University of Strassburg conditions which will be attractive for more advanced American graduate students of mathematics. Except at the University of Paris the department of mathematics will have the largest number of instructors of any university in France. For the coming year these are: PÈRÈS, professor of general mathematics; VALIRON, professor of differential and integral calculus; VILLAT, professor of mechanics; FRÉCHET, professor of higher analysis; ESCLARGON, professor of astronomy; and three maîtres de conferences, ANTOINE, DARNOIS, and VÈROMET.

Monsieur CHAZY has been appointed professor of general mathematics at the University of Lille, in place of Professor JEAN CLAIRIN, killed in battle in 1914.

CLAUDE GUICHARD, professor of general mathematics at the University of Paris, has been appointed to the chair of higher geometry there, made vacant by Darboux's death. Professor E. P. J. VESSIOT, of the University of Lyons, has been appointed to succeed Professor Guichard as professor of general mathematics.

Monsieur ROY has been appointed professor of rational mechanics at the University of Toulouse, to replace the late Professor LATTÈS.

Professor UGO AMALDI, of the University of Modena, has been appointed professor of descriptive geometry at the University of Padua.

Professor TULLIO LEVI-CIVITA, of the University of Padua, has been appointed professor of higher analysis at the University of Rome.

Professor CARLO SEVERINI, of the University of Catania, has been appointed professor of infinitesimal analysis at the University of Genoa.

A. MOHRMANN has been appointed ordinary professor of mathematics at the University of Basel.

RODOLPHE GUIMARÃES, the Portuguese engineer, died in 1918 aged 52 years. His 160-page work on *Les Mathématiques en Portugal au XIX^e Siècle*, prepared for the Paris Exposition in 1900, was enlarged by about 500 pages in the second edition (Coïmbre, 1909) entitled *Les Mathématiques en Portugal*.

MATTEO BOTTASSO, chargé du cours of rational mechanics at the University of Messina, died at Turin, October 3, 1918, aged 40 years. He was the author of *Analyse vectorielle générale*, volume 4: *Astatique* (Pavia, 1915), in the work of Buralli-Forti and Marcolongo.

Dr. C. BRANDENBERGER, professor of mathematics in the canton school, Zürich, and professor of mathematical didactics and methodology in the normal section of the Polytechnikum, died January 2, 1919, in the forty-sixth year of his age. He contributed, to the International Commission on the Teaching of Mathematics, the long report (1912) on mathematical instruction in Swiss gymnasia and "real schools."

Professor MAXIME BÔCHER'S mathematical library has been purchased for the library at the Proving Ground, Aberdeen, Md.

The state legislature of California has established a Southern California branch of the University of California in connection with the Los Angeles State Normal School. Dr. E. C. MOORE, who is known to our readers as a philosopher who has taken active part in recent discussions on mathematics, will be the director of this branch in addition to his duties as president of the Normal School. Miss MYRTIE COLLIER who was head of the department of mathematics in the Normal School has been made head of that department in the new school. Mr. G. E. F. SHERWOOD, who was formerly associate professor of mathematics at the Colorado College of Mines, and who has been a graduate student at the University of Chicago during the past year, will take charge of the collegiate work in mathematics.

In the MONTHLY for September, 1916, reference was made to the last will and testament of Professor MITTAG-LEFFLER and his wife, with its generous provisions for the founding of a Mathematical Institute. It was originally planned that the Institute should not be put into operation until after Professor Mittag-Leffler's death. But Mr. Jourdain announces in the last issue of *Science Progress* that Professor Mittag-Leffler "has handed over a capital sum to the Academy of Sciences so that the activity of the Institute can begin at the present time on a modest scale. 'I already have,' writes Prof. Mittag-Leffler, 'two scholars endowed with travelling fellowships, and will send them to England as soon as circumstances permit.'"

FOURTH SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

The Association held its fourth summer meeting by invitation of the University of Michigan at Ann Arbor, Michigan, on Thursday, Friday and Saturday, September 4–6, 1919, in conjunction with and following the meetings of the American Mathematical Society and the American Astronomical Society. One hundred ninety persons were in attendance, including official representatives of three institutions and the following 91 members of the Association:

R. C. Archibald, Brown University.
G. N. Armstrong, Ohio Wesleyan University.
Norman Anning, University of Maine.
L. A. Bauer, Carnegie Institution.
Mrs. Ethelwynn R. Beckwith, College for Women, Western Reserve University.
W. S. Beckwith, Ohio Northern University.
W. W. Beman, University of Michigan.
Suzan R. Benedict, Smith College.
G. A. Bliss, University of Chicago.
Henry Blumberg, University of Illinois.
J. D. Bond, A. and M. College of Texas.
R. L. Borger, Ohio University.
J. W. Bradshaw, University of Michigan.
W. C. Brenke, University of Nebraska.
E. W. Brown, Yale University.
W. D. Cairns, Oberlin College.
A. L. Candy, University of Nebraska.
E. H. Clarke, Hiram College.
Myrtie Collier, Southern Branch, University of California.
G. H. Cresse, U. S. Naval Academy.
D. R. Curtiss, Northwestern University.
Marian E. Daniells, Iowa State University.
S. C. Davisson, Indiana University.
F. F. Decker, Syracuse University.
L. W. Dowling, University of Wisconsin.
John Eiesland, West Virginia University.
L. P. Eisenhart, Princeton University.
L. C. Emmons, Michigan Agricultural College.
Fay Farnum, Iowa State College.
Peter Field, University of Michigan.
B. F. Finkel, Drury College.
J. A. Foberg, Crane Junior College.
W. B. Ford, University of Michigan.
C. F. Gummer, Queen's University.
W. A. Hamilton, Beloit College.
E. R. Hedrick, University of Missouri.
G. W. Hess, Bethany College.
T. H. Hildebrandt, University of Michigan.
H. A. Howe, University of Denver.
E. V. Huntington, Harvard University.
W. J. Hussey, University of Michigan.
H. R. Kingston, University of Manitoba.
A. E. Lampen, Hope College.
Florence P. Lewis, Goucher College.
G. H. Ling, University of Saskatchewan.
A. C. Lunn, University of Chicago.
E. B. Lytle, University of Illinois.
W. D. MacMillan, University of Chicago.
J. L. Markley, University of Michigan.
John Matheson, Queen's University.
J. V. McKelvey, Iowa State College.
G. A. Miller, University of Illinois.
E. J. Moulton, Northwestern University.
F. R. Moulton, University of Chicago.
A. L. Nelson, University of Michigan.
C. A. Nelson, University of Kansas.
H. L. Olson, University of Wisconsin.
F. W. Owens, Cornell University.
C. I. Palmer, Armour Institute.
A. D. Pitcher, Adelbert College.
L. C. Plant, Michigan Agricultural College.
R. G. D. Richardson, Brown University.
H. L. Rietz, University of Iowa.
Maria M. Roberts, Iowa State College.
E. D. Roe, Jr., Syracuse University.
W. H. Roever, Washington University.
J. R. Sage, Jr., Iowa State College.
G. T. Sellew, Knox College.
J. B. Shaw, University of Illinois.
H. E. Slaught, University of Chicago.
E. R. Sleight, Albion College.
G. W. Smith, University of Kentucky.
P. F. Smith, Yale University.
G. G. Speeker, Michigan Agricultural College.
W. M. Steirnagle, Jonesboro, Ark.
R. P. Stephens, University of Georgia.
E. B. Stouffer, University of Kansas.
A. L. Underhill, University of Minnesota.
J. N. Van der Vries, U. S. Chamber of Commerce.
H. E. Webb, Central High School, Newark, N. J.
R. A. Wells, Park College.
Mary L. Welton, Ann Arbor High School.